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AERODYNAMIC THEORY OF THE OSCILLATING WING OF FINITE SPAN

By

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PART I
GENERAL

1 Purpose

The theory of the oscillating airfoil of infinite span with two-dimensional flow has been developed to a high degree of completeness (ref. 4, 7)

In the present theory the oscillating airfoil of finite span is considered with the purpose of introducing the effect, of the trailing vortices and the three dimensional character of the velocity field, on the aerodynamic forces. General formulae are established for the lift and moment on an elliptic airfoil oscillating in both translation and rotation and the numerical results are presented in tabular and graphical form.

2 Summary

The plan form of the wing is thought of as elliptical (Fig. 1) While the methods developed here are not restricted to this case they are believed to apply best to elliptic wings. The aerodynamic lift and moment on the airfoil are evaluated for both translatory oscillations and rotary oscillations about the major axis DD' of the ellipse. Numerical results are presented for aspect ratios ranging from one to infinity.

The method is based on the linearized thin airfoil theory as developed originally by Glauert and Munk for the two dimensional problem and corrected for three dimensional aspect ratio effects by introducing the downwash induced by an appropriate distribution of free and bound vorticity. This vorticity distribution is not known exactly but it is possible to assume from the start an approximate pattern for the wake vorticity.

Another simplification is introduced by computing the aerodynamic lift and moment per unit span only at the mid span EF. It is then assumed that the total force on the wing will be found by integrating along the span using at every section the same expression as found for the mid span. It is seen that for a stationary wing with elliptic plan form this method will lead to the elliptic distribution alongthe span. The same method must also give approximately the total lift and moment for a wing which is not of elliptic plan form.

Notwithstanding these simplifications the problem as such would lead to analytical difficulties in the evaluation of the various integrals which occur during the development of the theory. These difficulties are avoided by the use of simple approximations for the downwash distribution

induced by the vorticity. In this way the integrands are replaced by approximate functions and the integrals are easily evaluated analytically.

As pointed out above it is sufficient to evaluate the lift and moment per unit span at the mid-span. This is given by the formulae 6 of Part I page 11. These expressions contain two complex quantities which are characteristic functions of the aspect ratio $A\!\!R$ and the "reduced velocity" $\frac{U}{\omega c}$ (U velocity of air stream, c maximum chord, ω angular frequency of the oscillations). These functions are denoted by

$$\bar{P}_{A\!R} = F_{A\!R} + i\,G_{A\!R}$$
$$\bar{Q}_{A\!R} = H_{A\!R} + i\,J_{A\!R}$$

These complex functions are plotted in figure 3 as vectors in the complex plane with $\frac{U}{\omega c}$ and $A\!\!R$ as variables. The real and imaginary parts are tabulated in tables 1 to 4 and plotted in figures 4 to 11 as functions of $\frac{U}{\omega c}$ for the aspect ratios,

$$A\!\!R = 1,\ 1\tfrac{1}{2},\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 10,\ 12,\ \infty\ .$$

Plots against $A\!\!R$ are also given in figure 12 to 15. The functions $\bar{P}_{A\!R}$ and $\bar{Q}_{A\!R}$ summarize completely the effect of the shed and trailing vortices on the aerodynamic forces. For the wing of infinite aspect ratio the functions become $\bar{P}_{A\!R} = \bar{Q}_{A\!R} = C$ where C is the function introduced by Theodorsen (reference 4).

3 Introduction

The theory of the non stationary airfoil has been the object of many investigations since the original work of W. Birbaum, H. Wagner, and H. Glauert. (ref. 8, 9, 10). The complete theory of the oscillating airfoil with a control surface was developed simultaneously by H. G. Kussner and T. Theodorsen. (ref. 4, 7) for an airfoil of infinite aspect ratio. The flow around the wing is two dimensional and the airfoil-flap system has three degrees of freedom, namely arbitrary translation and rotation and an arbitrary oscillation of the flap about the hinge. Considering that each degree of freedom is given an arbitrary oscillatory motion of angular frequency ω the problem is to find the lift and moment of the aerodynamic forces on the wing and the hinge moment on the flap. The essential difference of this case with a stationary airfoil is due to the existence of "shed vorticity" in the wake. The existence of vorticity in the wake for two dimensional flow is a consequence of the fact that for an oscillating airfoil the total circulation about the wing is variable. Vorticity must therefore be shed in the wake in order to satisfy the fundamental theorem of conservation of vorticity. It is the downwash induced by this wake vorticity on the airfoil which is cause that the forces on an oscillating airfoil are quite different from those derived from the stationary wing theory.

Until the present time the two dimensional theories have been applied as such to wings of finite aspect ratio. It is evident from what is known for the stationary wing that the forces acting on a wing of finite aspect ratio can be considerably different from those derived from the two dimensional theory. This difference is due to the influence of the so called trailing vortices which originate around the tips of the wing and lie in the wake along the direction of the wind velocity. They are perpendicular to the shed vorticity lines and do not vanish like the latter for the stationary case ($\omega = 0$).

The question therefore comes up of evaluating the aerodynamic forces on an oscillating airfoil of finite aspect ratio and it is seen in the light of the above remarks that this amounts essentially to evaluating the effect of the downwash induced simultaneously by what is referred to here as the shed and trailing vorticity.

In the present report only two degrees of freedom are considered. A translation of the wing h and rotation α of the wing about a spanwise axis through the mid chord. The problem is formulated more precisely by considering the harmonic translation and rotation expressed by the complex quantities

$$h = \bar{h} e^{i\omega t}$$

$$\alpha = \bar{\alpha} e^{i\omega t}$$

The lift and moment are then also complex quantities of the same type

$$L = \bar{L} e^{i\omega t}$$

$$M = \bar{M} e^{i\omega t}$$

It is important to point out that the methods used here are those of the linearized thin airfoil theory where the angle of attack and thickness of the wing are assumed to be small. In such a case the principle of superposition is valid and, as far as the forces due to the oscillations are considered, any simultaneous stationary twist, camber, or angle of attack has no effect. The oscillatory forces are the same as those of an uncambered, untwisted wing, performing oscillations about a position of zero angle of attack.

Under the assumption of small oscillations the forces may be considered to be linear functions of the coordinates h and α. Hence the general expressions for lift and moment.

$$L = L_h h + L_\alpha \alpha$$

$$M = M_h h + M_\alpha \alpha$$

The problem consists in the evaluation of the four "aerodynamic coefficients" L_h, L_α, M_h, M_α. These coefficients are complex quantities functions of the wind velocity and the frequency of the oscillation. In the case of a wing of finite aspect ratio and of definite plan form, say elliptic, they will also depend on the aspect ratio. The evaluation of L_h has been attempted elsewhere (reference 11) for a few aspect ratios but to the writers knowledge no values are at present available for any of the four aerodynamic coefficients within the complete practical range of the parameters.

In view of the complexity of the problem it is clear that a practical solution must necessarily involve a number of simplifying approximations. Therefore in the present theory the evaluation of the aerodynamic forces is limited to the lift and moment per unit span at the mid-span. It is then assumed that the forces on the complete wing are obtained by integration along the span evaluating the forces at every section by using the expression for the forces at the mid-span. Also certain assumptions are introduced regarding the distribution pattern of the trailing vortices. Although this is not essential, the plan form of the wing will be thought of as elliptical because the above assumptions are believed to hold best in this case.

With this simplified method it is now sufficient to compute the lift and moment per unit span on the mid span chord. The problem is thereby reduced to a two dimensional problem in the plane of symmetry of the wing and the procedure is entirely analogous to the known treatment of the wing of infinite aspect ratio except that the downwash on the mid span chord is due to trailing and certain correcting vortices in addition to the shed vorticity.

Another feature of the present treatment is the use of an artifice in order to perform the integrations which occur at various stages of the theory. The integrals are of such a nature that they could not be evaluated analytically or numerically unless perhaps with a prohibitive amount of labor. The artifice which has been used is to approximate the integrand by a simple function in such a way that the integration offers no difficulty. By this method it has been possible to use as such in the present theory the well known integrals expressible in terms of Bessels function which occur in the treatment of the wing of infinite aspect ratio. Additional integrals due to the finite character of the span appear in separate terms and do not involve any advanced method of integration. Therefore the mathematical difficulties encountered in the present treatment are exactly of the same nature as those of the now familiar two dimensional theory.

4 Basic Procedures and Assumptions

A detailed exposition of the theoretical procedure is given in part II of this report. A short account of these procedures will now be given in order to bring out the essential points.

The simplifying assumptions result from considering the wing to be equivalent to the following system illustrated in figure 2.

1. A flat plate P of infinite span whose motion is the same as that of the wing of finite span.
2. Shed vorticity S distributed sinusoidally in the wake
3. Trailing vorticity BC, $B'C'$, concentrated on two lines with a sinusoidally variable intensity.
4. Tip trailing vortices AB, $A'B'$, of uniform intensity.
5. A bound vortex located at the quarter chord and extending to infinity on each side of the wing AA_1, $A'A_1'$, with an intensity equal and opposite to the total circulation about the wing.

Once the total vorticity about the wing at the mid span is known the intensity of the vorticity in the assmmed vorticity pattern is completely determined.

The distance $2r_1$ between the trailing vortices depends on the aspect ratio and must clearly be determined by an additional assumption. The assumption introduced is that for a stationary airfoil, the downwash at the quarter chord AA' (Fig. 2) determined from the lifting line theory with elliptic distribution is the same as that due to the concentrated trailing vortices ABC and $A'B'C'$. Denoting the aspect ratio by AR this gives for the distance (see Part II Chapter I Section 2)

$$2r_1 = \frac{c\,AR}{2}$$

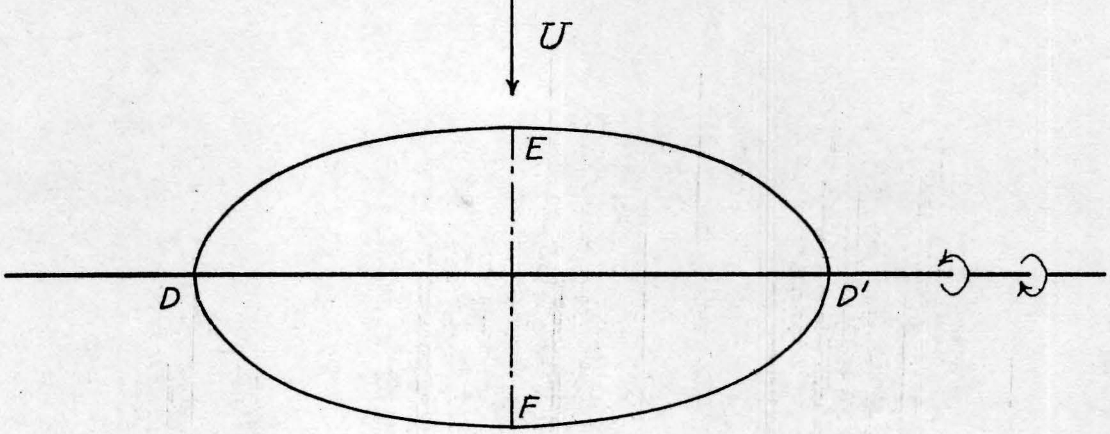


Fig. 1 - Elliptic Plan Form

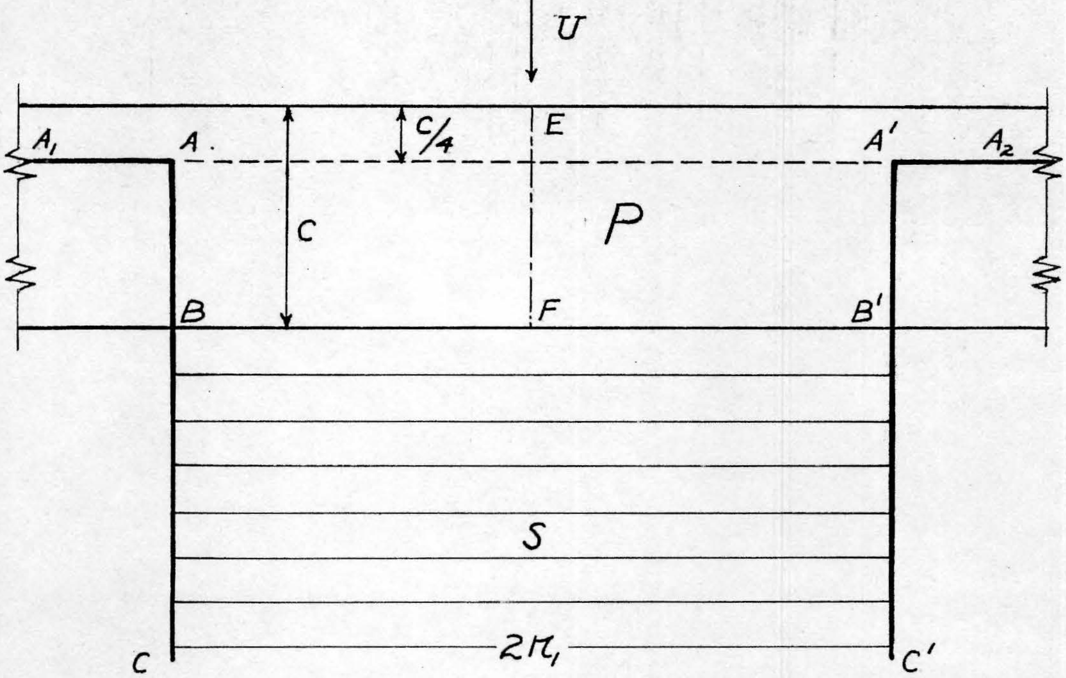


Fig. 2 - System of Infinite Wing and Vortex Pattern Used to Simplify the Problem of a Wing of Finite Span

c is the chord. If the plan form of the wing is elliptical c is taken as the maximum chord.

The pattern of vortices is now completely determined except for a common intensity factor which is the unknown total circulation about the airfoil.

As mentioned in the introduction it is sufficient to compute the aerodynamic forces on the chord EF at the mid span. This may be considered as a two dimensional problem in the plane of symmetry of the wing. All the known formulae of the two dimensional thin airfoil theory may then be applied provided the downwash distribution along EF, due to the assumed three dimensional vorticity pattern, is taken into account.

The first step is to determine the unknown total circulation A about the airfoil. This is done by applying Munk's integral

$$A = -c\int_0^{\pi} w\,(1+\cos\tau)\,d\tau \tag{1}$$

as set forth in part II chapter II. The variable of integration is related to the abscissa x along the chord EF by the relation

$$x = \frac{c}{2}\cos\tau$$

and the origin is taken at the mid chord. In this integral w is a "downwash distribution" along EF and may therefore be taken as a function of τ. It can be separated into two parts and written

$$w = w_0 + w_v$$

w_0 is the instantaneous normal component of the velocity of the undisturbed air relative to the wing; it is completely known when the motion of the wing is given.

w_v is the downwash due to the assumed vorticity pattern shown in figure 2.

Since this vorticity pattern contains the total circulation A as an unknown factor, the downwash w_v contains this same factor and equation (1) in therefore a linear equation which determines the value of A

Once A is determined the downwash w is completely known.

Using the two dimensional thin airfoil theory it is possible to derive expressions of the lift L and moment M (about the mid chord) per unit span which depend only on w. This is done in part II chapter III section 1. The results are,

$$L = \rho U A + \frac{\rho c}{2} \frac{\partial A}{\partial t} - \rho \frac{\partial B_2}{\partial t} \tag{2}$$

$$M = -\rho U B_2 - \frac{\rho c^2}{16} \frac{\partial A}{\partial t} + \frac{\rho}{2} \frac{\partial B_3}{\partial t} \tag{3}$$

where

$$B_2 = \frac{c^2}{2} \int_0^\pi w \sin^2 \tau \, d\tau \tag{4}$$

$$B_3 = \frac{c^3}{4} \int_0^\pi w \sin^2 \tau \cos \tau \, d\tau \tag{5}$$

and ρ = mass of air per unit volume

U = wind velocity

c = chord

t = time

M positive when stalling

The evaluation of the integrals (1), (4), (5) is made practical by the use of approximating functions for the downwash distribution w. These approximations are introduced in part II chapter I sections 3, 4, 5.

5 Results

The values of the lift and moment per unit span at the mid span are derived from expressions (2), (3), (4), (5). The derivation is carried out in detail in part II chapter III sections 2, 3. The results expressed by equation (3.7) and (3.9) of that chapter are

$$\begin{aligned} L &= \pi\rho c U \dot{h}\left[\frac{i\omega c}{4U} + \bar{P}_{AR}\right] \\ &\quad + \pi\rho c U^2 \alpha\left[\frac{i\omega c}{4U} + \left(1+\frac{i\omega c}{4U}\right)\bar{P}_{AR}\right] \\ M &= \frac{\pi}{4}\rho c^2 U \dot{h}\, \bar{Q}_{AR} \\ &\quad + \frac{\pi}{4}\rho c^2 U^2 \alpha\left[\frac{\omega^2 c^2}{32 U^2} - \frac{i\omega c}{4U} + \left(1+\frac{i\omega c}{4U}\right)\bar{Q}_{AR}\right] \end{aligned} \tag{6}$$

where L = lift per unit span

M = moment about mid chord per unit span, positive when stalling.

ρ = mass of air per unit volume

U = wind velocity

c = mid span chord

ω = angular frequency of oscillation (radians/sec)

$\dot{h} = i\omega\bar{h}e^{i\omega t}$ complex downward velocity of the wing.

$\alpha = \bar{\alpha}e^{i\omega t}$ complex angle of attack of wing

$\bar{P}_{AR}$ and $\bar{Q}_{AR}$ are complex quantities functions of the "reduced velocity" $\frac{U}{\omega c}$ and the aspect ratio AR. Separating real and imaginary parts the the following notation is adopted

$$\bar{P}_{AR} = F_{AR} + i\,G_{AR}$$
$$\bar{Q}_{AR} = H_{AR} + i\,J_{AR}$$

For infinite aspect ratio ($AR=\infty$) the functions $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ are equal and in this case

$$F_\infty = H_\infty = F$$
$$G_\infty = J_\infty = G$$

where F and G denote the well known functions derived by Theodorsen.

The values of the functions $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ are given in tabular form in tables 1 to 4. In figure 3 the functions $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ are plotted as vectors in the complex plane. The functions F_{AR} G_{AR} H_{AR} and J_{AR} are plotted against $\frac{U}{\omega c}$ in figures 4 to 11 for the following values of the aspect ratio

$$AR = 1, 1\tfrac{1}{2}, 2, 3, 4, 5, 6, 8, 10, 12, \infty .$$

These same quantities are also plotted against the aspect ratio AR for different values of $\frac{U}{\omega c}$ in figure 12 to 15.

The range of $\frac{U}{\omega c}$ in these tables and charts is from .1 to 10

Expressions (6) give the forces on the wing per unit span at the mid span. It is assumed that the total forces for the whole wing are obtained by integrating expression (6) along the span. This process is sometimes referred to as the "strip method". In order to simplify the integration $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ are taken as independent of the location along the span and are given a constant value corresponding to the mid-span chord In this way $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ may be factorized out of the integral sign.

TABLE 1

Real Part of $\bar{P}_{A\!R}$; i.e., $Rl\,\bar{P}_{A\!R} = \bar{F}_{A\!R}$

The Numbers Heading the Columns Refer to Aspect Ratio

λc	$U/\omega c$	1	$1\frac{1}{2}$	2	3	4	5	6	8	10	12	∞
0.00	∞	0.25128	0.34475	0.42184	0.53827	0.61930	0.67758	0.72101	0.78083	0.81982	0.84713	1.00000
0.02	50	0.25035	0.34304	0.41932	0.53435	0.61436	0.67195	0.71495	0.77445	0.81351	0.84110	0.98242
0.05	20	0.24917	0.34101	0.41657	0.53076	0.61064	0.66857	0.71211	0.77262	0.81215	0.83954	0.95434
0.1	10	0.24765	0.33882	0.41417	0.52899	0.60985	0.66816	0.71119	0.76752	0.80360	0.82668	0.90901
0.2	5	0.24587	0.33747	0.41392	0.52957	0.60742	0.65974	0.69622	0.74240	0.77004	0.78840	0.83192
0.3	3-1/3	0.24711	0.33793	0.41482	0.52645	0.59696	0.64290	0.67451	0.71473	0.73897	0.75481	0.77280
0.7	1-3/7	0.24405	0.33641	0.40619	0.49799	0.55319	0.58840	0.61267	0.63986	0.65194	0.65676	0.64290
1.0	1	0.24135	0.33065	0.39627	0.48215	0.53348	0.56520	0.58475	0.60354	0.60964	0.61105	0.59794
2	0.5	0.22302	0.30976	0.37343	0.45595	0.50005	0.52275	0.53446	0.54400	0.54681	0.54752	0.53944
5	0.2	0.14576	0.25636	0.33765	0.42938	0.47076	0.49099	0.50171	0.51130	0.51474	0.51599	0.50874
10	0.1	0.011055	0.18928	0.29983	0.40758	0.45922	0.48210	0.49499	0.50613	0.51013	0.51156	0.50240
200	.005	-2.09286	-0.74505	-0.24990	0.18635	0.36044	0.44236	0.48495	0.52210	0.53469	0.53866	0.49997

TABLE 2

The Negative of the Imaginary Part of $\overline{P}_{A\!R}$; i.e., $-Im\ \overline{P}_{A\!R} = -G_{A\!R}$

The Numbers Heading the Columns Refer to Aspect Ratio

λc	$U/\omega c$	1	$1\frac{1}{2}$	2	3	4	5	6	8	10	12	∞
0.00	∞	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.000000	0.000000	0.000000	0.0000000
0.02	50	0.0029095	0.0032839	0.0039531	0.0055295	0.0070162	0.0083018	0.0094056	0.011229	0.012745	0.014299	0.045652
0.05	20	0.0058797	0.0056054	0.0058666	0.0079014	0.010293	0.012923	0.015696	0.021439	0.027111	0.032404	0.087239
0.1	10	0.0098220	0.0077520	0.0073536	0.0099033	0.015380	0.022400	0.029922	0.043719	0.055655	0.064620	0.13064
0.2	5	0.016480	0.010838	0.0099580	0.018199	0.032506	0.047072	0.059845	0.079271	0.092784	0.10279	0.17230
0.3	3-1/3	0.022969	0.014830	0.015120	0.029318	0.047764	0.063994	0.077197	0.096885	0.11122	0.12259	0.18646
0.7	1-3/7	0.052636	0.037577	0.038068	0.051246	0.066495	0.080808	0.093905	0.11614	0.13287	0.14477	0.17231
1.0	1	0.074020	0.052989	0.048139	0.054030	0.066411	0.080216	0.093298	0.114013	0.12740	0.13567	0.15071
2	0.5	0.14296	0.090687	0.066166	0.053562	0.059466	0.068410	0.075980	0.085734	0.090931	0.093867	0.10027
5	0.2	0.30510	0.16622	0.10125	0.055626	0.043781	0.040272	0.039288	0.039411	0.040135	0.040867	0.047297
10	0.1	0.48222	0.24827	0.14684	0.071330	0.039314	0.27814	0.021546	0.017096	0.015999	0.015959	0.024598
200	0.005	2.44850	1.25959	0.74433	0.30569	0.13467	0.055057	0.013940	-0.021683	-0.033604	-0.037257	0.0012599

TABLE 3

Real Part of $\bar{Q}_{\mathcal{A}}$; i.e., $Rl\,\bar{Q}_{\mathcal{A}} = H_{\mathcal{A}}$

The Numbers Heading the Columns Refer to Aspect Ratio

λc	$U/\omega c$	1	$1\frac{1}{2}$	2	3	4	5	6	8	10	12	∞
0.00	∞	0.35092	0.43375	0.49638	0.58878	0.65437	0.70295	0.74009	0.79266	0.82783	0.85291	1.00000
0.02	50	0.34963	0.43160	0.49342	0.58450	0.64914	0.69711	0.73387	0.78618	0.82146	0.84686	0.98242
0.05	20	0.34802	0.42907	0.49021	0.58057	0.64521	0.69359	0.73093	0.78430	0.82006	0.84525	0.95434
0.1	10	0.34602	0.42641	0.48744	0.57864	0.64435	0.69312	0.72992	0.78033	0.81134	0.83221	0.90901
0.2	5	0.34397	0.42506	0.48739	0.57932	0.64152	0.68416	0.71429	0.75323	0.77716	0.79339	0.83192
0.3	3-1/2	0.34365	0.42623	0.48886	0.57595	0.63044	0.66641	0.69167	0.72478	0.74541	0.75921	0.77280
0.7	1-3/7	0.34805	0.42878	0.48147	0.54530	0.58354	0.60908	0.62672	0.64726	0.65538	0.65924	0.64290
1.0	1	0.35118	0.42635	0.47271	0.52873	0.56242	0.58376	0.59695	0.60903	0.61208	0.61195	0.59794
2.	0.5	0.35744	0.42058	0.45858	0.50378	0.52673	0.53771	0.54278	0.54620	0.54624	0.54553	0.53944
5	0.2	0.37512	0.42836	0.45756	0.48572	0.49755	0.50328	0.50636	0.50918	0.51025	0.51066	0.50874
10	0.1	0.40172	0.442263	0.46197	0.47954	0.49013	0.49482	0.50130	0.50089	0.50222	0.50286	0.50240
200	0.005	0.47152	0.45359	0.45261	0.46160	0.47106	0.47822	0.48335	0.48969	0.49316	0.49520	0.49997

TABLE 4

The Negative of the Imaginary Part of $\bar{Q}_{AR}$!; i.e., $-Im\ \bar{Q}_{AR} = -J_{AR}$

The Numbers Heading the Columns Refer to Aspect Ratio

λc	$U/\omega c$	1	$1\frac{1}{2}$	2	3	4	5	6	8	10	12	∞
0.00	∞	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.000000	0.000000	0.000000	0.000000
0.02	50	0.0016041	0.0027735	0.0038838	0.0058338	0.0074237	0.0087245	0.0098143	0.011590	0.013061	0.014376	0.045652
0.05	20	0.0020702	0.0036630	0.0050175	0.0081025	0.0108964	0.013683	0.016508	0.022240	0.027853	0.033079	0.087239
0.1	10	0.0014538	0.0029804	0.0048111	0.0097489	0.016291	0.023788	0.031503	0.045759	0.057135	0.065947	0.130644
0.2	5	-0.0014586	0.0001136	0.0040538	0.017753	0.034384	0.049882	0.062924	0.082236	0.095435	0.105104	0.172302
0.3	3-1/3	-0.0047041	-0.0015737	0.0063573	0.028846	0.050490	0.067823	0.081311	0.100805	0.114736	0.12572	0.186456
0.7	1-3/7	-0.013006	0.0012511	0.019025	0.048444	0.069561	0.086086	0.099964	0.12229	0.13838	0.14911	0.172314
1.0	1.0	-0.015923	0.0027000	0.020931	0.048311	0.068723	0.085674	0.099903	0.12078	0.13357	0.14117	0.150709
2	0.5	-0.023341	-0.0025822	0.013567	0.038849	0.058627	0.072860	0.082466	0.093087	0.097962	0.100110	0.100273
5	0.2	-0.040145	-0.016820	0.0006391	0.022890	0.034481	0.040678	0.044156	0.047460	0.048742	0.049263	0.047297
10	0.1	-0.048247	-0.018212	-0.0015562	0.012788	0.020862	0.023676	0.025349	0.026509	0.026795	0.026805	0.024598
200	0.005	-0.014682	-0.024772	0.034286	0.032852	0.026684	0.021218	0.017057	0.011607	0.008508	0.006605	0.001260

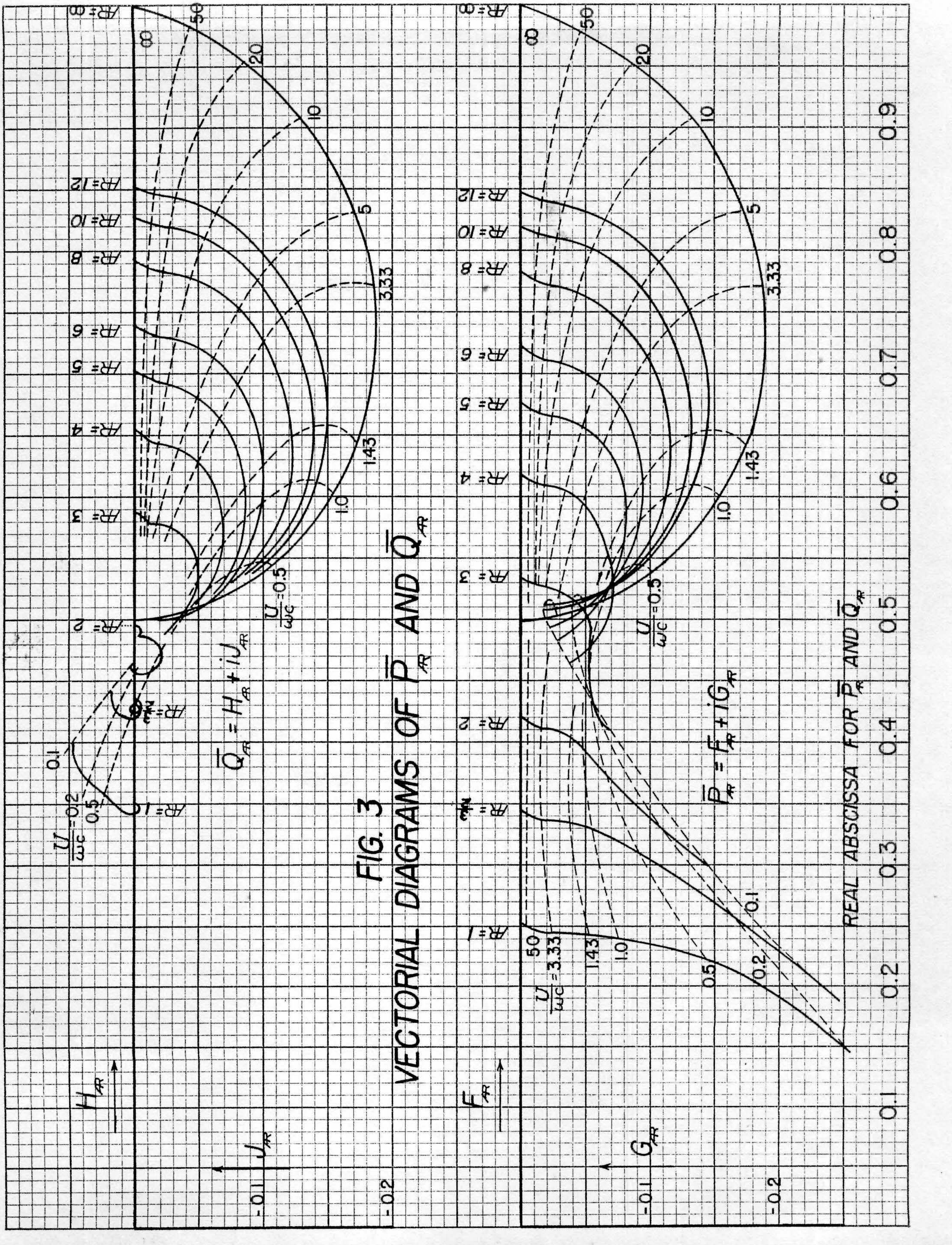
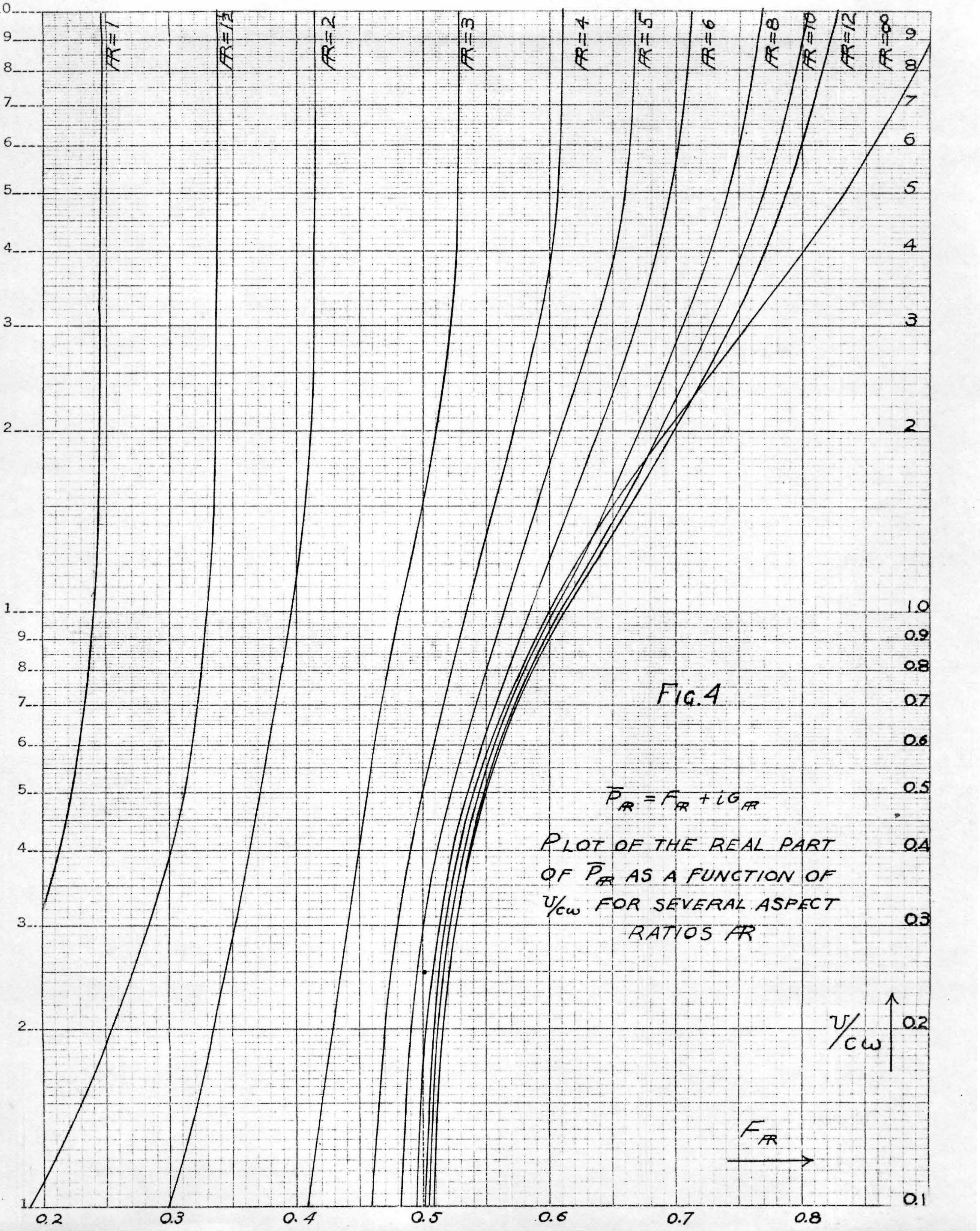


FIG. 3

VECTORIAL DIAGRAMS OF $\overline{P}_{AR}$ AND $\overline{Q}_{AR}$


AR=1
AR=1½
AR=2
AR=3
AR=4
AR=5
AR=6
AR=8
AR=10
AR=12
AR=∞
FIG. 4
$\bar{P}_{AR} = F_{AR} + iG_{AR}$
PLOT OF THE REAL PART OF $\bar{P}_{AR}$ AS A FUNCTION OF $U/c\omega$ FOR SEVERAL ASPECT RATIOS AR
$U/c\omega$
F_{AR}
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2
3
4
5
6
7
8
9

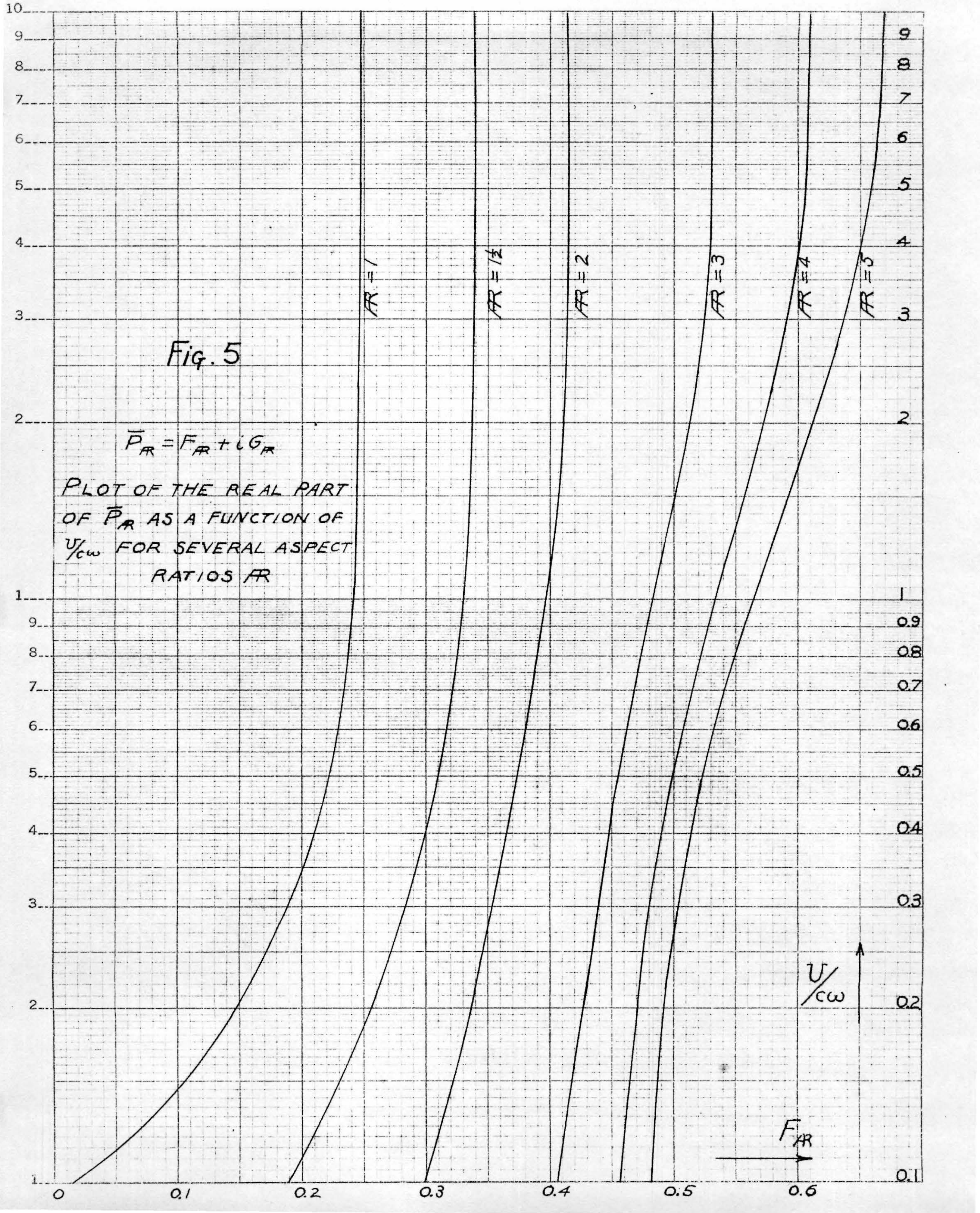


Fig. 5
$\bar{P}_{AR} = F_{AR} + iG_{AR}$
PLOT OF THE REAL PART OF $\bar{P}_{AR}$ AS A FUNCTION OF $U/c\omega$ FOR SEVERAL ASPECT RATIOS AR
AR = 1
AR = 1½
AR = 2
AR = 3
AR = 4
AR = 5
$U/c\omega$
F_{AR}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
2
3
4
5
6
7
8
9

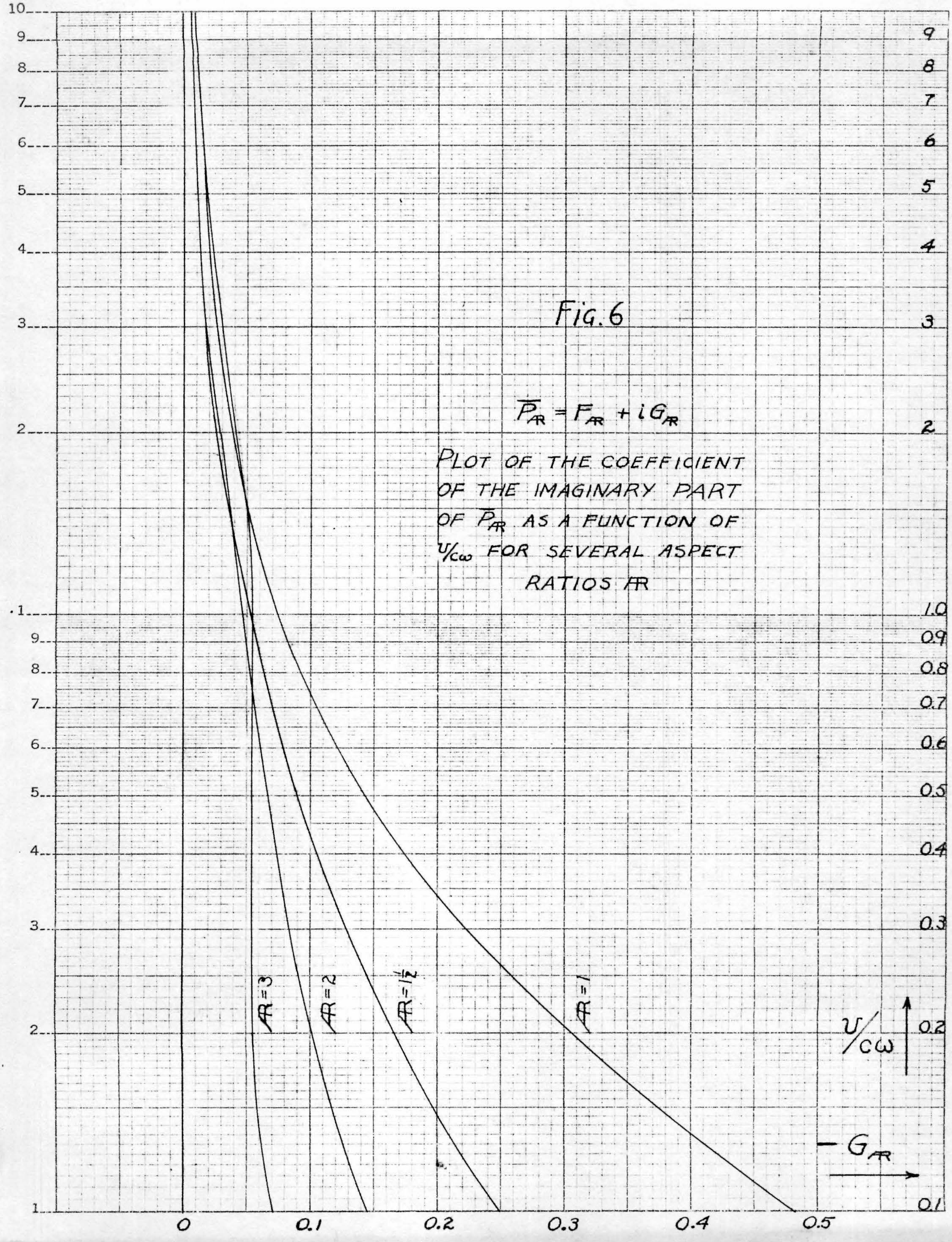


Fig. 6
$\overline{P}_{AR} = F_{AR} + iG_{AR}$
PLOT OF THE COEFFICIENT OF THE IMAGINARY PART OF $\overline{P}_{AR}$ AS A FUNCTION OF $v/c\omega$ FOR SEVERAL ASPECT RATIOS AR
AR=3
AR=2
AR=1½
AR=1
$v/c\omega$
$-G_{AR}$
0
0.1
0.2
0.3
0.4
0.5
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
9
8
7
6
5
4
3
2

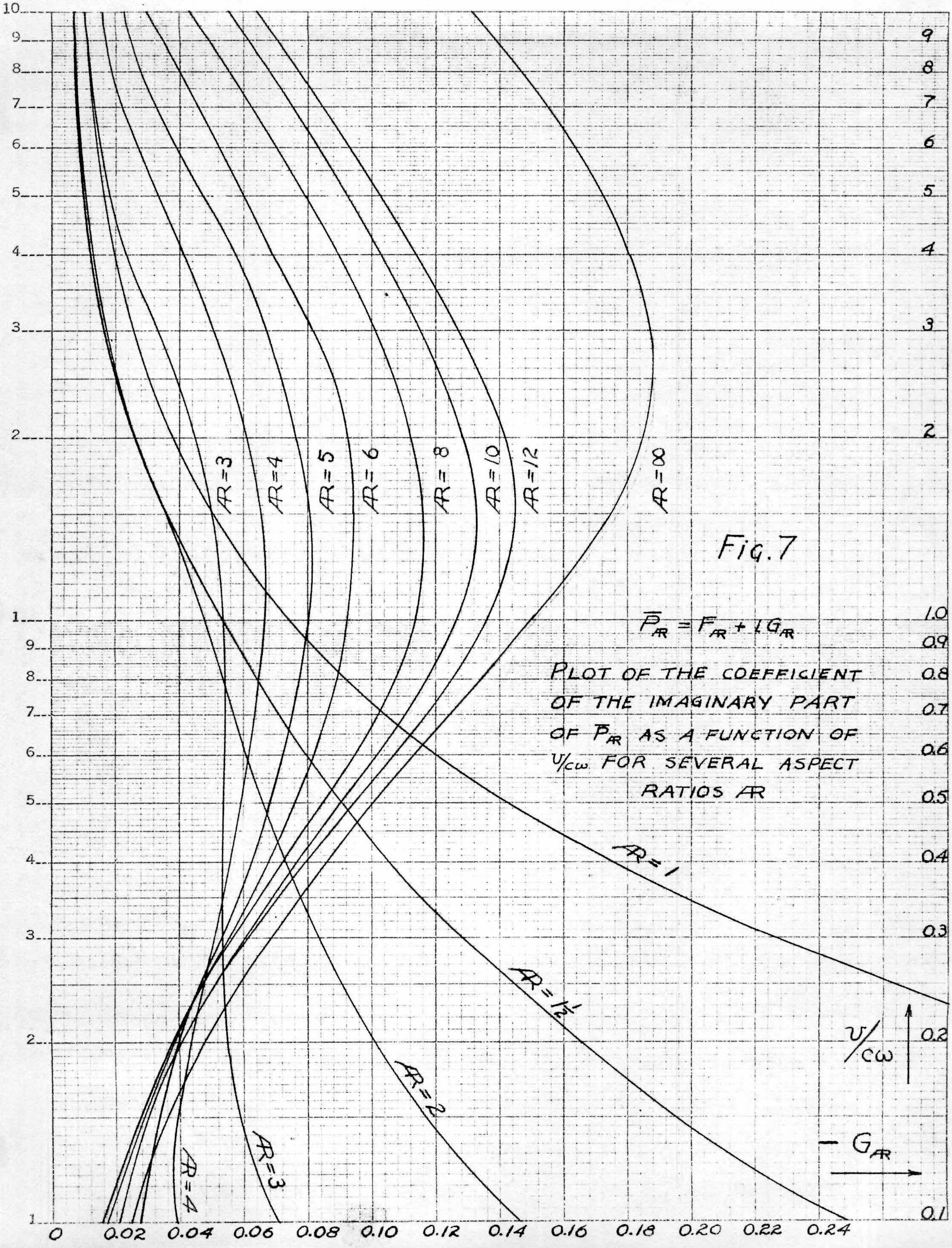


Fig. 7
$\bar{P}_{AR} = F_{AR} + iG_{AR}$
PLOT OF THE COEFFICIENT OF THE IMAGINARY PART OF $\bar{P}_{AR}$ AS A FUNCTION OF $v/c\omega$ FOR SEVERAL ASPECT RATIOS AR
AR=1
AR=1½
AR=2
AR=3
AR=4
AR=5
AR=6
AR=8
AR=10
AR=12
AR=∞
$v/c\omega$
$-G_{AR}$
0
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2
3
4
5
6
7
8
9

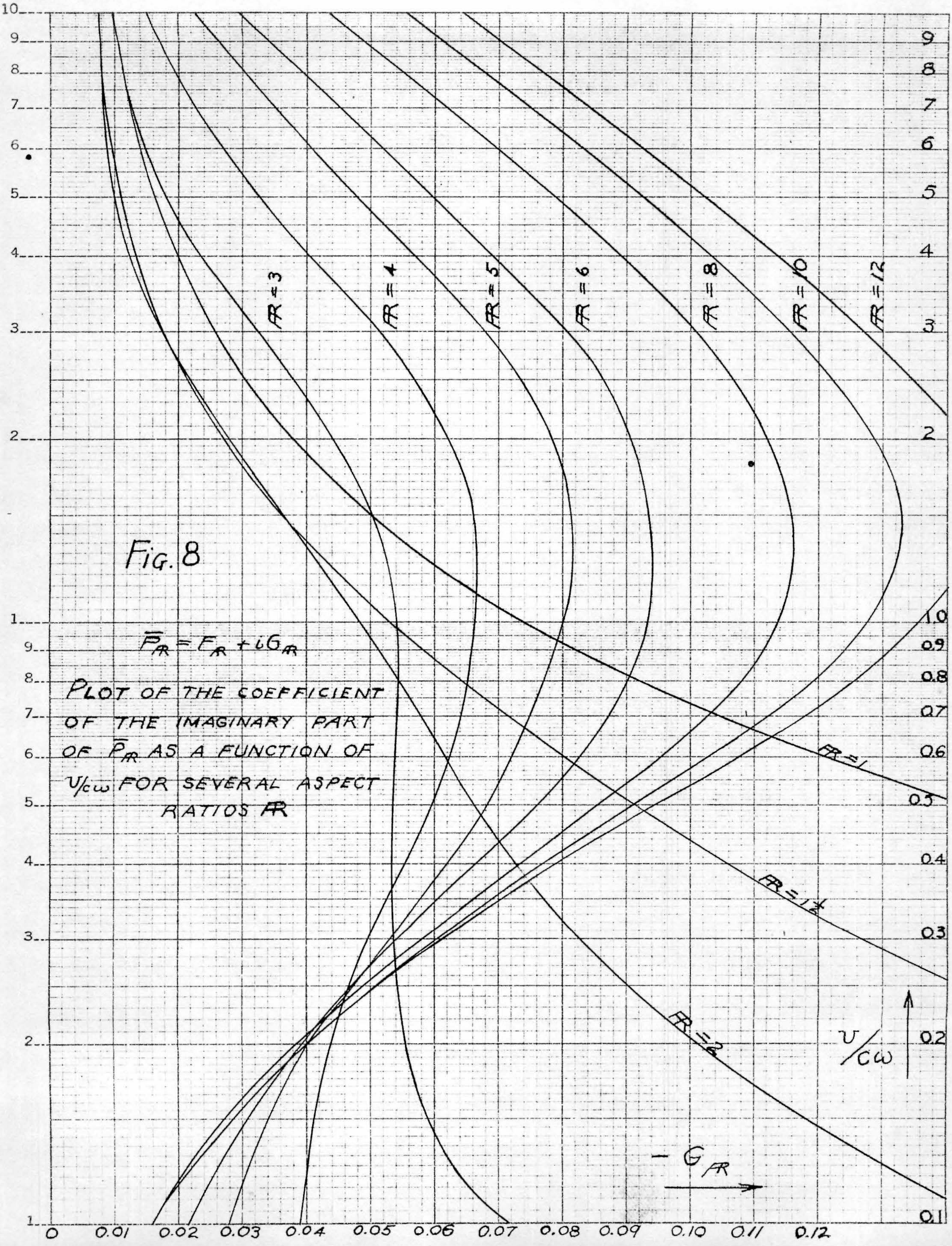


FIG. 8
$\bar{P}_{AR} = F_{AR} + iG_{AR}$
PLOT OF THE COEFFICIENT OF THE IMAGINARY PART OF $\bar{P}_{AR}$ AS A FUNCTION OF $v/c\omega$ FOR SEVERAL ASPECT RATIOS AR
AR = 1
AR = 1½
AR = 2
AR = 3
AR = 4
AR = 5
AR = 6
AR = 8
AR = 10
AR = 12
$-G_{AR}$
$v/c\omega$
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
0.10
0.11
0.12
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2
3
4
5
6
7
8
9

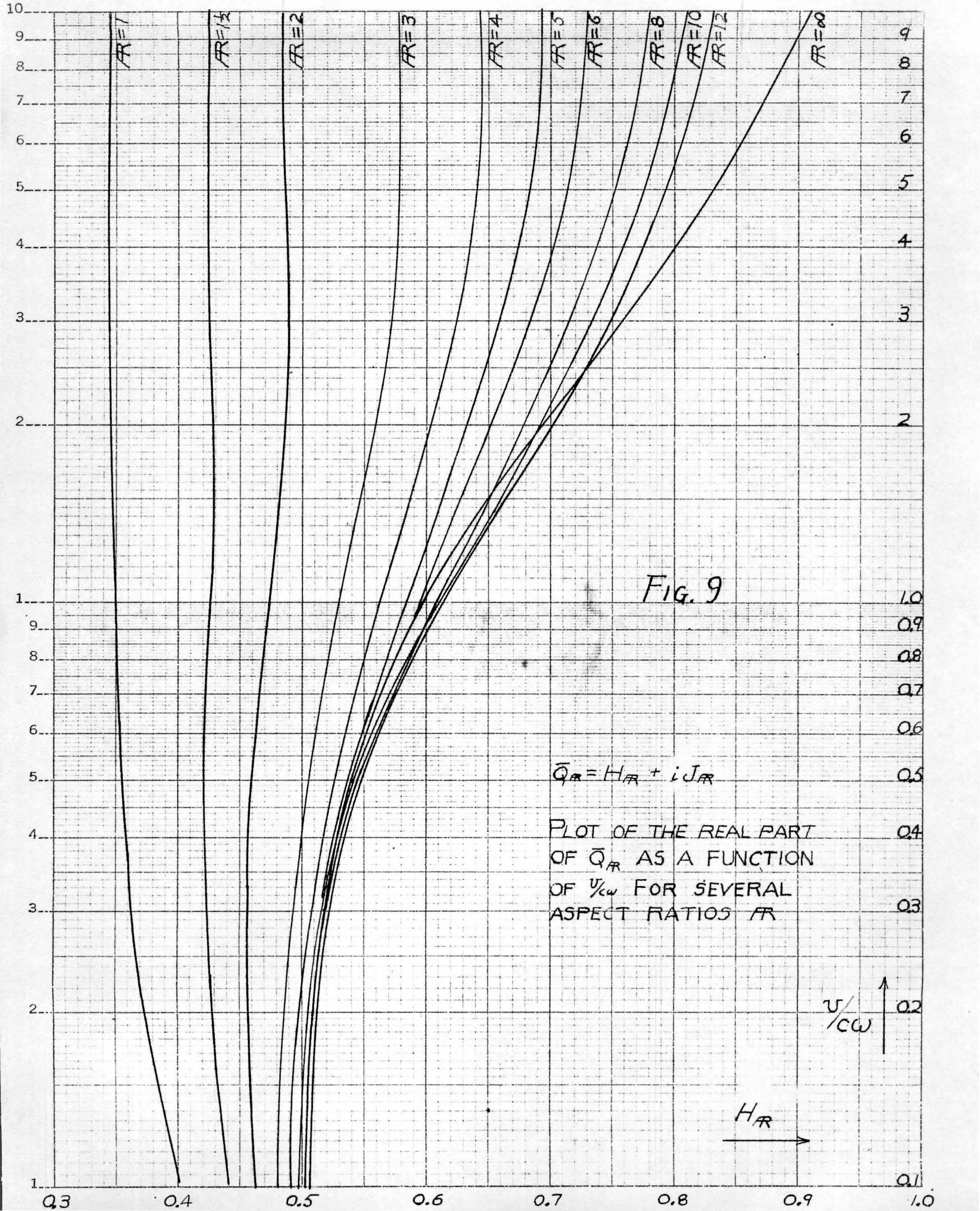


FIG. 9

$\bar{Q}_{AR} = H_{AR} + i J_{AR}$

PLOT OF THE REAL PART OF $\bar{Q}_{AR}$ AS A FUNCTION OF $v/c\omega$ FOR SEVERAL ASPECT RATIOS AR

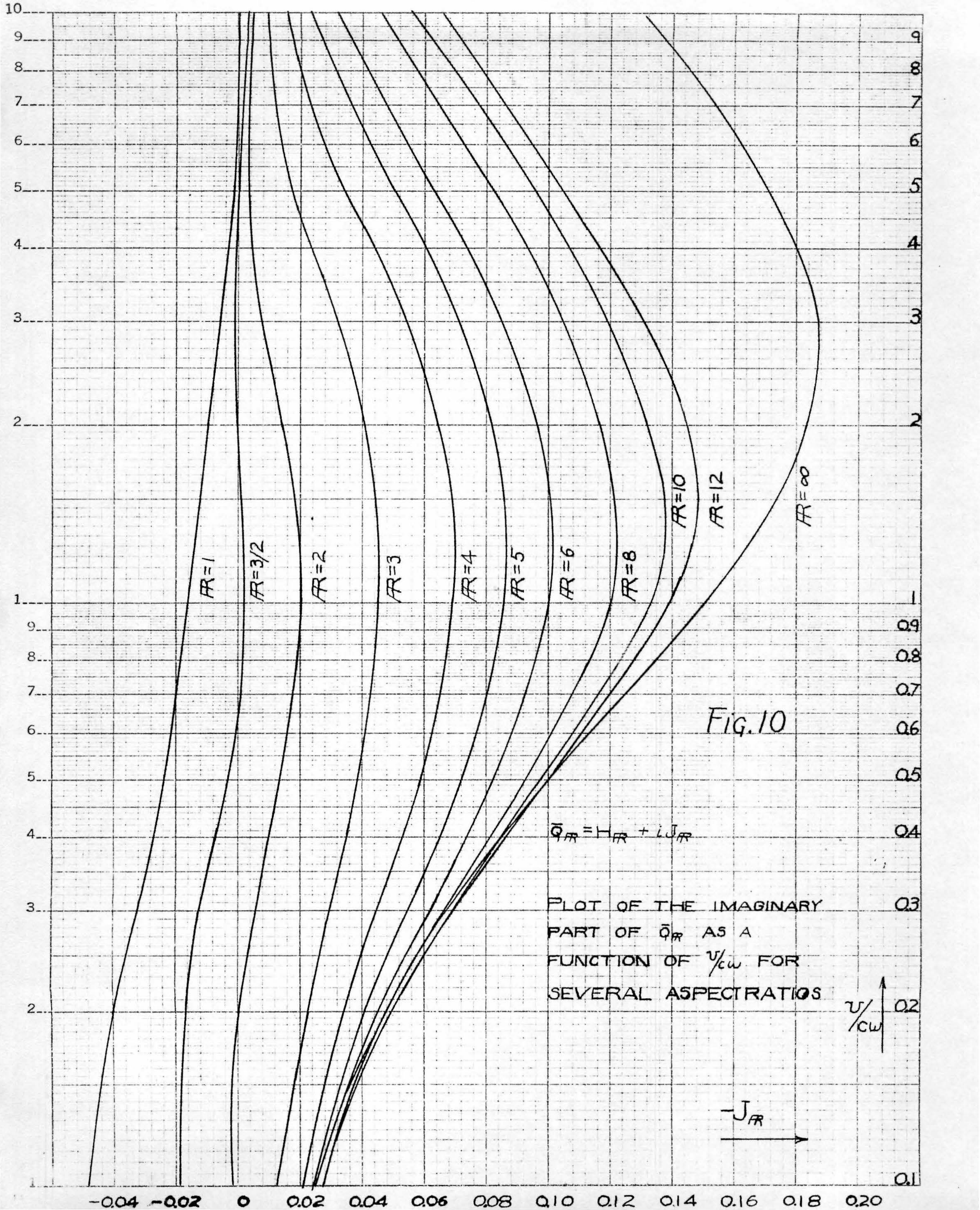


FIG. 10

PLOT OF THE IMAGINARY PART OF $\bar{Q}_{AR}$ AS A FUNCTION OF $v/c\omega$ FOR SEVERAL ASPECT RATIOS

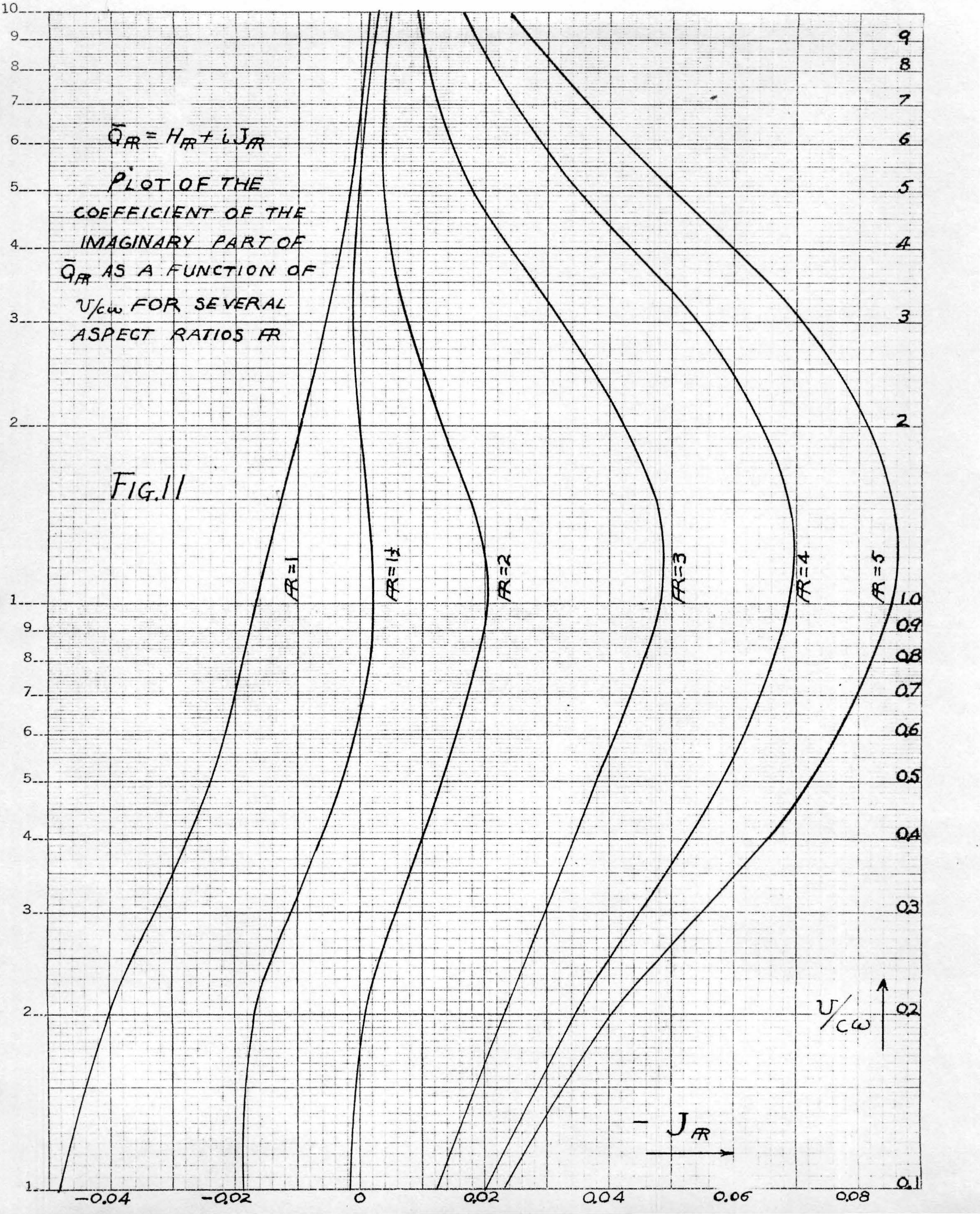

$\bar{Q}_{AR} = H_{AR} + i J_{AR}$
PLOT OF THE COEFFICIENT OF THE IMAGINARY PART OF $\bar{Q}_{AR}$ AS A FUNCTION OF $U/c\omega$ FOR SEVERAL ASPECT RATIOS AR
$AR=1$
$AR=1\frac{1}{2}$
$AR=2$
$AR=3$
$AR=4$
$AR=5$
$U/c\omega$
$-J_{AR}$
-0.04
-0.02
0
0.02
0.04
0.06
0.08
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
2
3
4
5
6
7
8
9

FIG. 11

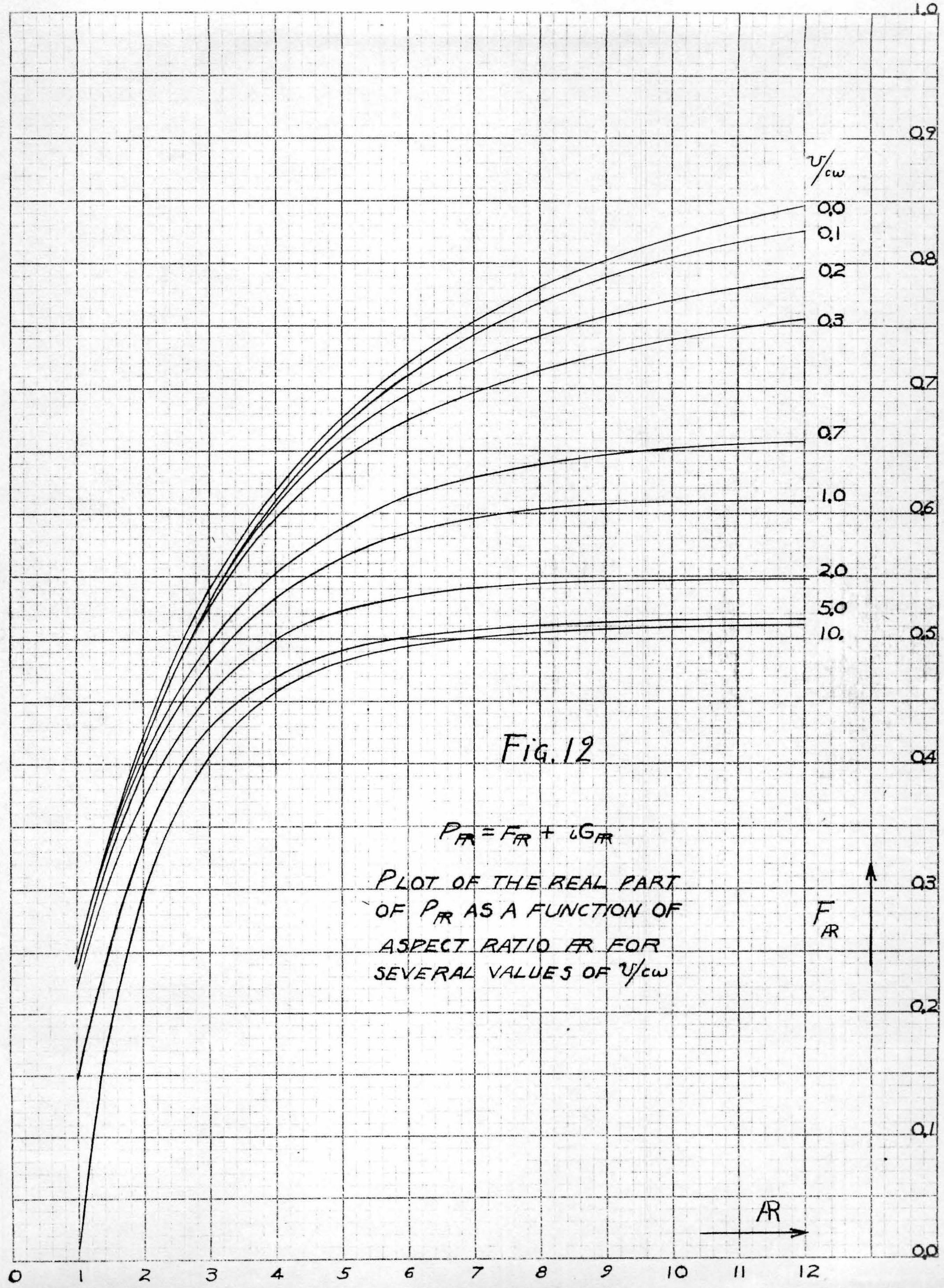


FIG. 12

$P_{AR} = F_{AR} + iG_{AR}$

PLOT OF THE REAL PART OF P_{AR} AS A FUNCTION OF ASPECT RATIO AR FOR SEVERAL VALUES OF $v/c\omega$

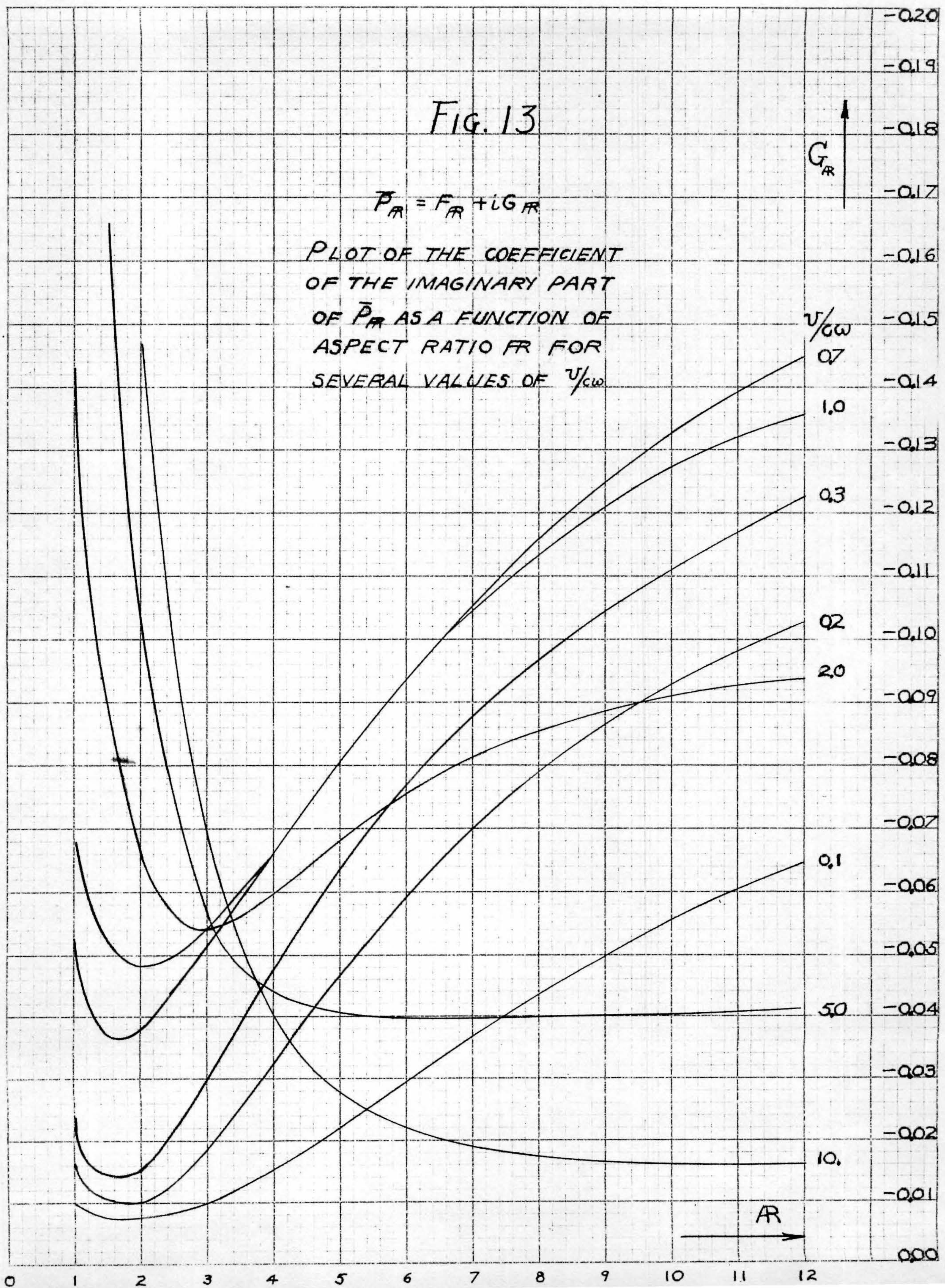


FIG. 13
$\bar{P}_{AR} = F_{AR} + iG_{AR}$
PLOT OF THE COEFFICIENT OF THE IMAGINARY PART OF $\bar{P}_{AR}$ AS A FUNCTION OF ASPECT RATIO AR FOR SEVERAL VALUES OF $v/c\omega$
G_{AR}
$v/c\omega$
0.7
1.0
0.3
0.2
2.0
0.1
5.0
10.
AR
-0.20
-0.19
-0.18
-0.17
-0.16
-0.15
-0.14
-0.13
-0.12
-0.11
-0.10
-0.09
-0.08
-0.07
-0.06
-0.05
-0.04
-0.03
-0.02
-0.01
0.00
0
1
2
3
4
5
6
7
8
9
10
11
12

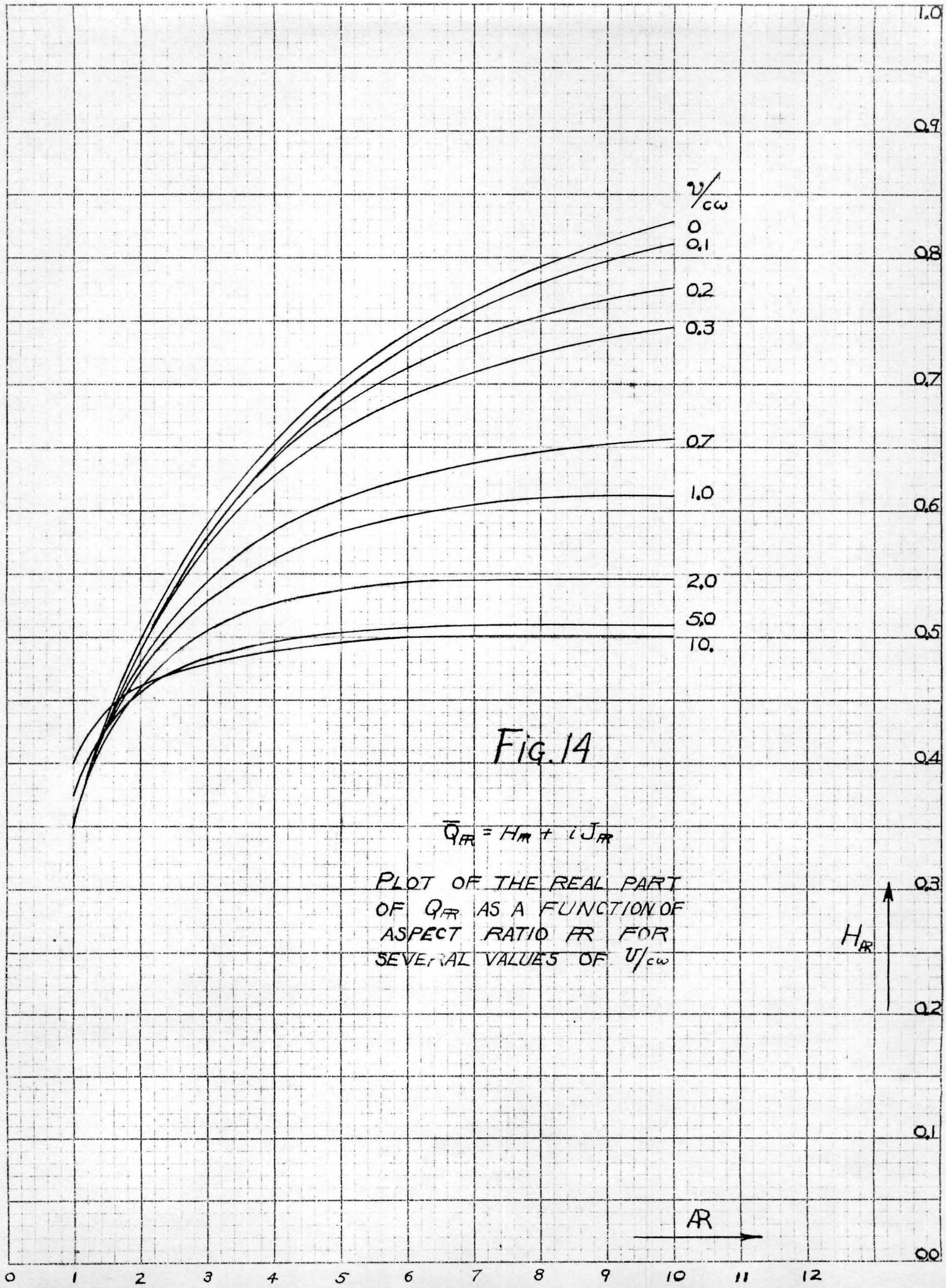


FIG. 14

$$\bar{Q}_{AR} = H_{AR} + iJ_{AR}$$

PLOT OF THE REAL PART OF Q_{AR} AS A FUNCTION OF ASPECT RATIO AR FOR SEVERAL VALUES OF $v/c\omega$

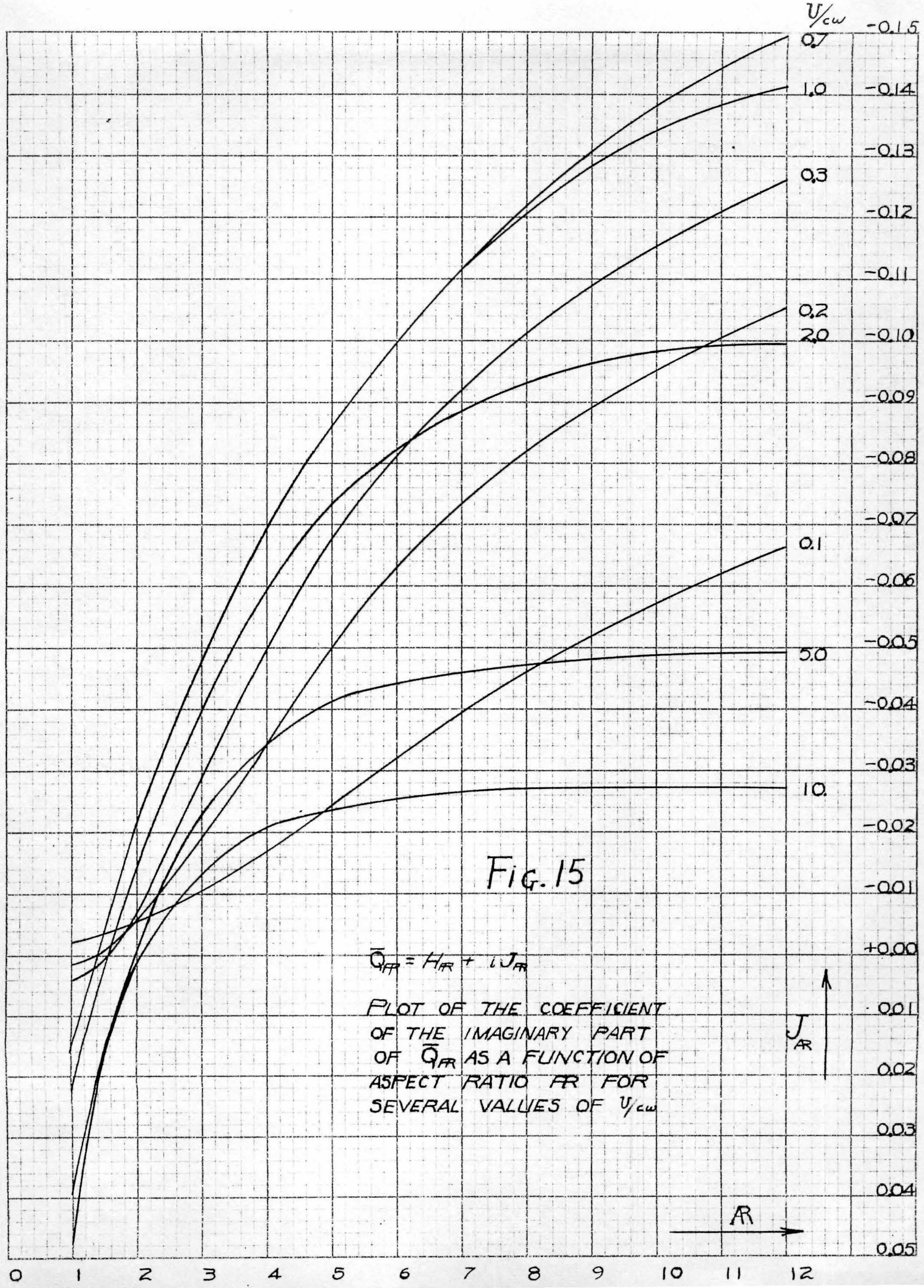
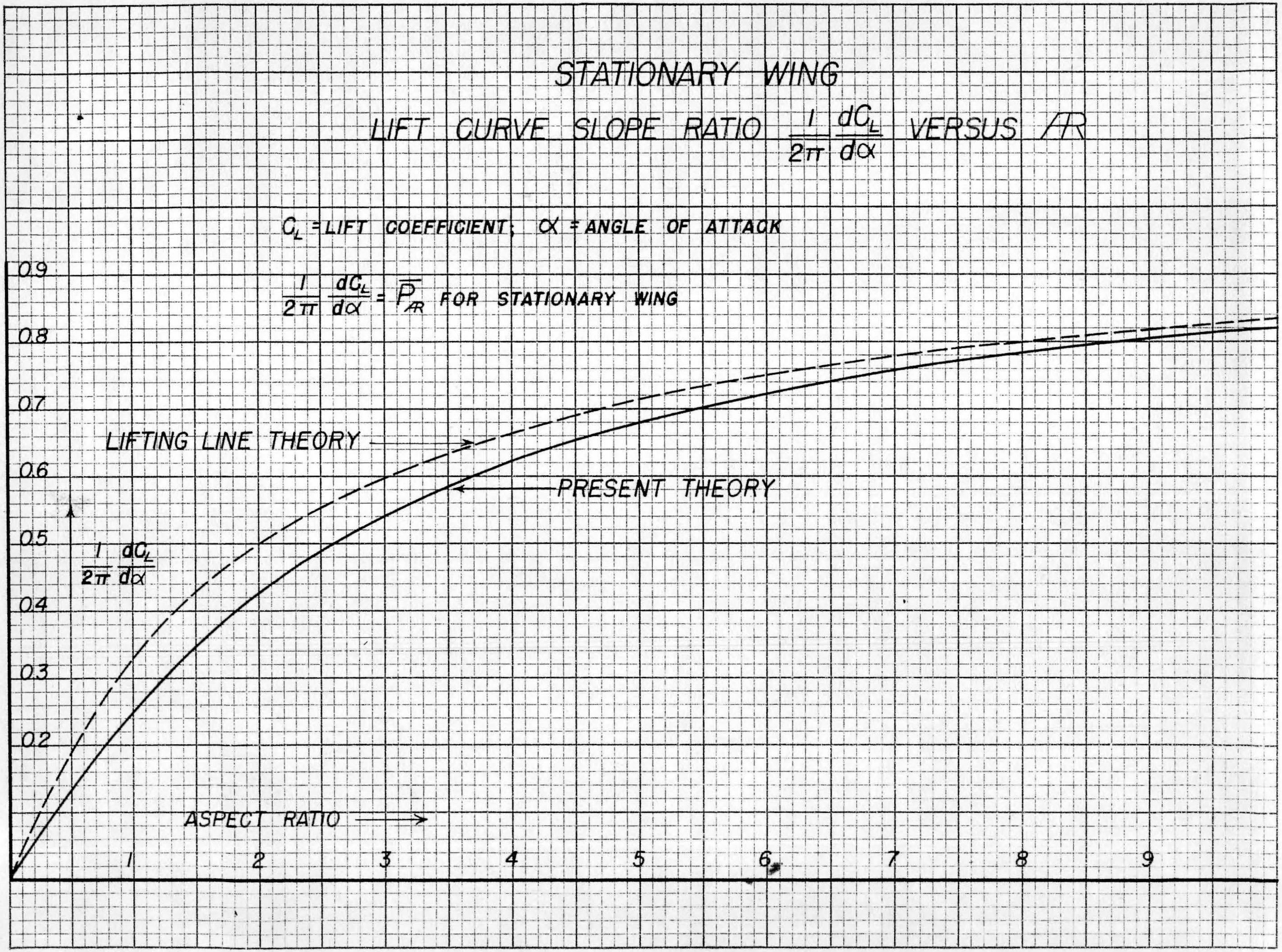


FIG. 15

PLOT OF THE COEFFICIENT OF THE IMAGINARY PART OF $\bar{Q}_{AR}$ AS A FUNCTION OF ASPECT RATIO AR FOR SEVERAL VALUES OF $U/c\omega$


STATIONARY WING
LIFT CURVE SLOPE RATIO $\frac{1}{2\pi}\frac{dC_L}{d\alpha}$ VERSUS $Æ$
C_L = LIFT COEFFICIENT; α = ANGLE OF ATTACK
$\frac{1}{2\pi}\frac{dC_L}{d\alpha} = \overline{P}_{Æ}$ FOR STATIONARY WING
LIFTING LINE THEORY
PRESENT THEORY
$\frac{1}{2\pi}\frac{dC_L}{d\alpha}$
ASPECT RATIO
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
1
2
3
4
5
6
7
8
9

6. Discussion

The functions $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ are due to the existence of circulation about the wing. They represent in condensed form the influence of the trailing and shed vorticity on the aerodynamic forces. This is expressed mathematically by the functional dependence of $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ on AR and $\frac{U}{\omega c}$

It is possible to express the forces given by (6) in the following form

$$L = L_1 + L_2$$
$$M = M_1 + M_2$$

$$L_1 = \frac{\pi\rho c^2}{4}\ddot{h} + \frac{\pi}{4}\rho c^2 U\dot{\alpha}$$
$$M_1 = -\frac{\pi\rho c^4}{128}\ddot{\alpha} + \frac{\pi}{4}\rho c^2 U\dot{h} + \frac{\pi}{4}\rho c^2 U^2\alpha$$

$$L_2 = \pi\rho c U^2\left(\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha}c}{4U}\right)\bar{P}_{AR}$$
$$M_2 = \frac{\pi}{4}\rho c^2 U^2\left(\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha}c}{4U}\right)\left(\bar{Q}_{AR} - 1\right)$$

The forces L_1, M_1 are those which would occur for pure potential flow without the existence of circulation about the wing.i.e., when the Kutta-Joukowski condition is not satisfied at the trailing edge. The terms proportional to the accelerations $\ddot{h}$ and $\ddot{\alpha}$ are the inertia forces due to the so called apparent mass of the surrounding air. The forces L_2, M_2 are caused by the existence of circulation about the wing. The intensity of these forces depends on the aspect ratio AR and the reduced velocity $\frac{U}{\omega c}$ through the functions $\bar{P}_{AR}$ and $\bar{Q}_{AR}$.

It will be noticed that L_2, M_2 are proportional to $\alpha + \frac{\dot{h}}{U} + \frac{\dot{\alpha}c}{4U}$ which represent the effective instantaneous angle of attack at a point $\frac{3}{4}c$ from the leading edge (three-quarter chord)

For the stationary wing with infinite aspect ratio $\bar{P}_{AR} = \bar{Q}_{AR} = 1$ and the forces reduce to

$$L = \pi \rho c U^2 \alpha$$

$$M = \frac{\pi}{4} \rho c^2 U^2 \alpha$$

Since M is a moment about the mid chord, the forces are reduced to a single resultant $L = \pi \rho c U^2 \alpha$ located at a point $\frac{c}{4}$ from the leading edge (quarter chord).

For the stationary wing with finite aspect ratio

$$L = \pi \rho c U^2 \alpha \bar{P}_{AR}$$

$$M = \frac{\pi}{4} \rho c^2 U^2 \alpha \bar{Q}_{AR}$$

The resultant is located at a distance $\frac{c}{4} \frac{\bar{P}_{AR}}{\bar{Q}_{AR}}$ from the mid chord. From the charts it is seen that $\bar{P}_{AR}$ is approximately equal to $\bar{Q}_{AR}$ and therefore the "aerodynamic center" is approximately at the quarter chord. This holds fairly well down to aspect ratio three.

The value of $\bar{P}_{AR}$ at zero frequency as a function of the aspect ratio is plotted in figure 16. It represents the so called "slope of the lift curve" dependence on the aspect ratio. The curve derived by the present theory checks very well with the results obtained for the elliptical stationary wing by by K. Krienes and R. T. Jones (reference 12). The curve obtained from the lifting line theory is also shown for comparison.

When small values of $U/\omega c$ are considered it is seen that the dependence of $\bar{P}_{AR}$ and $\bar{Q}_{AR}$ on aspect ratio is much smaller than for the stationary wing. This property must play a definite part in reducing the dependence of flutter speed on aspect ratio.

The general aspect of the curves are similar for aspect ratios from infinity down to three. For lower aspect ratios than three the nature of the curves becomes quite different.

Because of the approximations regarding the nature of the vorticity pattern especially the tip trailing vortices the theory breaks down for a zero value of $\frac{U}{\omega c}$. It is believed however that the results computed are sufficiently accurate for practical purposes down to the value $\frac{U}{\omega c} = .1$ which correspond to a wave length in the shed vorticity equal to 63% of the chord

7 Conclusions

The forces on an oscillating airfoil of finite span have been derived theoretically. The dependence of these forces on the aspect ratio for the oscillating airfoil is smaller than for the stationary wing, especially at low values of the reduced velocity $\frac{U}{\omega c}$. At aspect ratios from infinity down to three the curves obtained show the same general trend. For aspect ratios lower than three the curves become of a quite different nature revealing a probable difference in the mechanism of the wing-vorticity interaction. It can be inferred that for flutter occuring at low values of $\frac{U}{\omega c}$ (say $\frac{U}{\omega c} < 1$) the effect of the finite span on the critical velocity will be small unless the aspect ratio is smaller than three.

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12. R. T. Jones; Theoretical Correction for the Lift of Elliptic Wings; Journal of the Aeronautical Sciences, Vol. 9, No. 1, page 8, November 1941

PART II

DETAILS OF PROCEDURE

Part II - Details of procedure (continued)

Summary of Symbols

Roman, Upper Case

A, instantaneous value of the total circulation about an oscillating wing.

$\bar{A}$, a complex constant: amplitude of the total circulation A.

AR, aspect ratio.

B, instantaneous value of the density of the shed vorticity at the trailing edge.

B_2, defines an integral concerning the downwash along the chord, introduced in section III-1 as expression (1.7).

$B_2', B_2'', B_2''', B_2^o$,
define integrals which are component parts of B_2, introduced in section III-2 as expressions (2.1), (2.2), (2.3), and (2.4) respectively. Their sum is B_2

B_3, defines an integral concerning the downwash along the chord, introduced in section III-1 as integral (1.12).

$B_3', B_3'', B_3''', B_3^o$,
define integrals which are component parts of B_3, introduced in section III-2 as expressions (2.6), (2.7), (2.8), and (2.9) respectively. Their sum is B_3.

$\bar{B}_2$, a complex constant: amplitude of B_2.

$\bar{B}_3$, a complex constant: amplitude of B_3.

F, a function of aspect ratio and $\frac{U}{\omega c} = \frac{1}{\lambda c}$ due to the fact that wing span is finite, introduced in section II-6 as expression (6.2).

F_T, F_S, F_C,
component parts of F due to the wake trailing, the shed, the bound and tip trailing vortices respectively. Their sum is F.

F_1 a function of aspect ratio and $\frac{U}{\omega c} = \frac{1}{\lambda c}$ due to the integration of B_2, introduced in section III-2 as expression (2.21).

F_{1T}, F_{1S}, F_{1C},
component parts of F_1 due to the wake trailing, the shed, the bound and tip trailing vortices respectively. Their sum is F_1.

F_0, a function of aspect ratio and $\frac{U}{\omega c} = \frac{1}{\lambda c}$ due to the integration of B_3, introduced in section III-2 as expression (2.35).

F_{0T}, F_{0S}, F_{0C},
component parts of F_0 due to the wake trailing, the shed, the tip and bound vortices respectively, Their sum is F_0.

F_{AR}, the real part of $\overline{P}_{AR}$, introduced in section III-3, expression (3.10).

H_{AR}, the real part of $\overline{Q}_{AR}$, introduced in section III-3, expression (3.11).

G_{AR}, the imaginary part of $\overline{P}_{AR}$, introduced in section III-3, expression (3.10).

J_{AR}, the imaginary part of $\overline{Q}_{AR}$, introduced in section III-3, expression (3.11).

K, used in section I-4 as a real constant, and in appendix B as the index of summation of series.

L, wing lift per unit of span.

M, wing moment per unit of span about an axis through the midpoint of the chord.

$\overline{P}_{AR}$, a function of aspect ratio and $U/\omega c = 1/\lambda c$, reducing to the C of Theodorson's work for infinite aspect ratio (see reference 4), introduced in section III-3 as expression (3.6).

$\overline{Q}_{AR}$, a function of aspect ratio and $\frac{U}{\omega c} = \frac{1}{\lambda c}$ reducing to the C of Theodorson's work for infinite aspect ratio (see reference 4), introduced in section III-3 as expression (3.8).

Q_0, a function of $\frac{U}{\omega c} = \frac{1}{\lambda c}$ only, composed of a linear coefficient, an exponential and a Hankel function of zero order, introduced in section II-3 as expression (3.3).

Q_1, a function of frequency only, composed of a linear coefficient, an exponential and a Hankel function of order one, introduced in section II-3 as expression (3.4).

U, velocity of the undisturbed air stream.

V same as U, used in section II-1 and appendix D.

S, wing area, used only in section I-2.

T, period for one cycle of the wing's oscillation used in section I-6 and appendix A.

Summary of Symbols (Cont'd)

Roman, Lower Case

a, represents radial distance in Biot-Savart law in section I-3 and appendix D. Used as the radius of the circle which is converted into an airfoil by conformal transformation of section II-1. Used as a constant in the integral $\int_0^\pi \frac{d\tau}{a-b\cos\tau}$ given in sections II-3 and III-2.

a_0, a_1, a_2, functions of aspect ratio, introduced section I-5 by expression (5.5).

b, used as the span of the elliptical wing in section I-2 and as a constant in the integral $\int_0^\pi \frac{d\tau}{a-b\cos\tau}$ given in sections II-3 and III-2.

c, chord.

c_K, designates the integral $\int_0^\pi \cos^{2K}\tau\, d\tau$, introduced in appendix B.

e, base of natural logarithms.

f, frequency of wing oscillations introduced in section I-6.

$\dot{h}$, instantaneous downward velocity of wing due to translatory oscillations introduced in section I-6 by expression (6.1).

i, the imaginary unit; i.e., $\sqrt{-1}$.

k, used as an auxiliary variable corresponding to $\frac{\lambda c}{2}$ in appendix C.

m, used as a constant in section I-4.

n, represents an integer in appendix B.

p, designates the unit pressure introduced in section III-1 expression (1.1).

p_1, p_2, designates the unit pressures acting on the upper and lower surfaces respectively of the wing, introduced in section II-1.

r_1, represents the semi-span of the hypothetical wing, introduced in section I-1, related to AR in that $r_1 = \frac{c\,AR}{4}$.

s, used in sections I-3 and I-4 to designate the expression $\frac{\xi + x}{r_1}$. Used in section I-5 to designate distance along the vortex line.

t, designates time.

u, in expression (1.1) section III-1 designates the x-component of velocity, in expression (1.2) designates perturbation velocity only. Also used to designate one of the variables in the formula for integrating by parts.

u_1, u_2, designates the x-component of velocity on the upper and lower surfaces respectively of the wing, introduced in section III-1.

v, in expression (1.1) section III-1 designates the y-component of velocity also used to designate one of the variables in the formula for integration by parts.

w, designates downwash velocity in general, designates the z-component of the velocity in expression (1.1) section III-1.

w_1, w_2, designates the downwash velocity due to the bound and tip trailing vortices; introduced in section I-5. Their sum is w'''.

w', w'', w''', downwash due to wake trailing, shed, tip and bound vorticity respectively.

w_T, downwash due to translatory oscillations of wing, introduced in section I-6, expression (6.1). w_R, downwash due to rotational oscillations.

$\overline{w}_T$, a complex constant; amplitude of the downwash due to translatory oscillations, introduced in section I-6 expression (6.2).

w_0, downwash due to the translatory and rotational oscillations, introduced in section I-6 as expression (6.6).

x, used as a variable of position on the chord. Used as a variable in general in appendices B and C.

y, ordinate of a thin airfoil in section II-1; variable in general in appendix B

Summary of Symbols (Cont'd)

Greek, Upper and Lower Case

α, instantaneous value of the angle of attack, introduced in section I-6.

$\dot{\alpha}$, instantaneous value of the angular velocity due to rotational oscillations, introduced in section I-6.

$\bar{\alpha}$, a complex constant: amplitude of the angle of attack, introduced in section I-6.

Γ, circulation in general, introduced in section II-1 expression (1.1).

Γ_0, circulation about midspan of elliptical wing, introduced in section I-2.

Γ^o circulation due to the translatory and rotational oscillations of wing, introduced in section II-5.

Γ', Γ'', Γ'''
circulation due to the wake trailing, shed, tip and bound vorticity respectively.

γ, vorticity along chord of wing, introduced in section III-1.

γ_1, value of the circulation at any point on the wake trailing vortices, introduced in section I-3.

γ_2, value of the vortex density (vorticity) of the shed vorticity, introduced in section I-4, expression (4.1).

ε, used as an infinitesimal in section III-1, see figure 1.1 .

ζ, non dimensional variable of integration, introduced in appendix C.

θ, used as the variable in connection with the Biot-Savart law in section I-3, I-4, and I-5. Used in section III-1 as an auxiliary in expression (1.6), related to x in that $x = \frac{c}{2}\cos\theta$ (origin at midpoint of chord).

θ_1, θ_2 specific values of θ used in connection with the Biot-Savart law in sections I-4 and I-5.

λ, equals 2π divided by the wave length, introduced in section I-3.

ν, used as the subscript of order on the Bessel functions in appendices B and C.

ξ, coordinate extending down stream from trailing edge of wing, introduced in section I-3.

ρ, density of the air.

τ, auxiliary variable, introduced in section II-1, see expressions (1.3) and (1.4).

ψ, ψ', variables used in integral (2D) of appendix D.

ω, angular frequency (rad/sec) of oscillating wing, introduced in section I-3.

VORTICITY PATTERN AND DOWNWASH

I-1 The Vortex System

An oscillating wing of infinite aspect ratio is shown in figure 1.1. The relative wind is designated by the vector U and line EF represents a chord. Behind the wing is drawn a set of lines parallel to the span which represent the shed vorticity. This shed vorticity is also of infinite span and moves down stream with the relative wind. If the wing continues to oscillate for a long time, this shed vorticity will extend a great distance down stream from the trailing edge of the wing, thereby forming a vortex sheet, infinite in two dimensions.

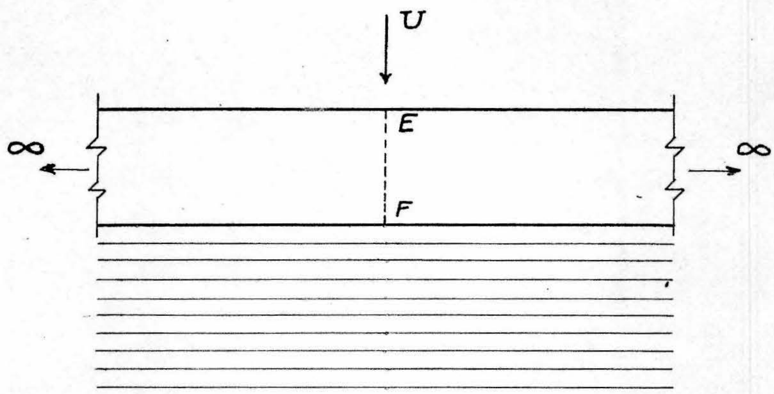

Fig. 1.1 - Wing of Infinite Aspect Ratio Showing Shed Vorticity

To simplify matters let the wing be replaced by a single bound vortex, called the lifting line, shown diagramatically in figure 1.2. . In this figure the wing is shown in phantom and the shed vorticity in full lines. It is to be noted that the lifting line is placed at a distance equal to one fourth the chord length C, from the leading edge and parallel to the span. The reason for

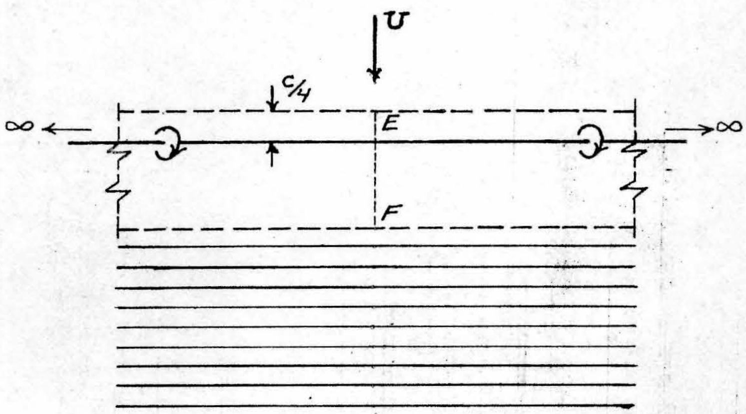

Fig. 1.2 - Wing Replaced by Lifting Line

placing the lifting line at this location is, that the quarter chord point is approximately the aerodynamic center for all wings both of infinite and finite span, and it is also the centroid of the bound vortex distribution.

In order to reduce the wing of infinite span to that of finite span the vortex pattern shown in figure 1.3 is superposed on figure 1.2. The vortex line extending from the point A, figure 1.3, to the left to infinity is taken with the same strength but with the opposite rotational sense to that of figure 1.2,

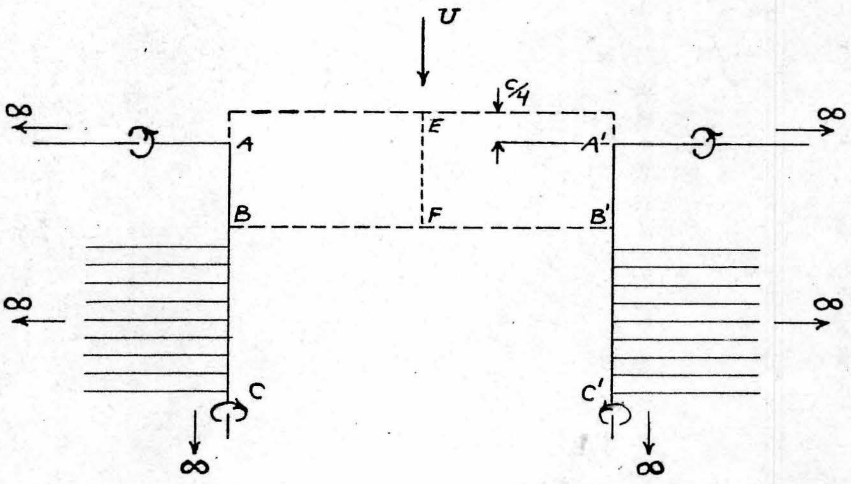

Fig. 1.3 - Superposed Vortex Pattern

as indicated by the curved arrows on figures 1.2 and 1.3. When the two figures are superposed these two vortices will nullify each other. The same can be said for the vortex line extending from the point A', figure 1.3, to the right to infinity; consequently all that remains of the lifting line is the segment between points A and A' as shown in figure 1.4. In figure 1.3 there is a vortex line extending from the point A through points B, C and thence to infinity. This vortex line is referred to as the <u>trailing vortex</u>. At the right hand side of figure 1.3 is another trailing vortex extending from point A' through B', C' and thence to infinity. The portion of vortex line from A to B is herein referred to as the <u>tip trailing</u>

<u>vortex</u> and that from B through C and thence to infinity as the <u>wake trailing vortex</u>; likewise for $A'B'$ and $B'C'$ to infinity. It is apparent that the trailing vortices will remain after superposition. Extending to the left of the wake trailing vortex BC, there is shown in figure 1.3 a set of lines which hypothetically extend to infinity. These lines represent a vorticity of such strength and rotational sense as to just cancel the shed vorticity to the left of the wake trailing vortex. The same condition is made to exist for the lines to the right of the wake trailing vortex $B'C'$ to infinity.

After superposition all that remains is the lifting line between the points A and A', shown in figure 1.4, the tip vortices AB and $A'B'$, the wake trailing vortices BC and $B'C'$, and that portion of the shed vorticity which lies between BC and $B'C'$. It is to be pointed out that the wake trailing vortices together with the shed vorticity form a vortex ribbon which extends from the trailing edge to infinity. The wing of figure 1.4 is shown in phantom and its span is designated as $2\Pi_1$. It is to be pointed out again that the lifting line AA' is at the quarter chord point, the line BB' represents the trailing edge, and the line EF is the midchord.

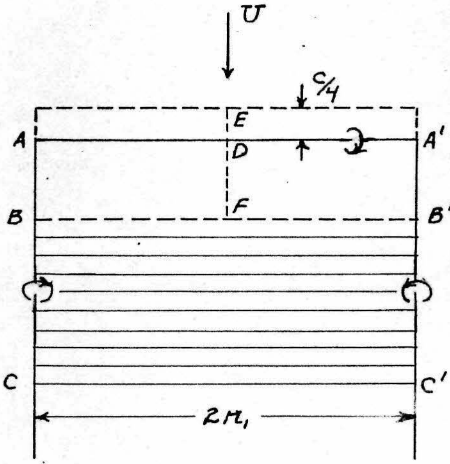

Fig. 1.4 - Final Vortex Pattern

In order to clarify the terminology which follows, another figure is shown on which is labeled the various terms used herein. This is figure 1.5. In this figure, the wing is shown in phantom, the chord length is taken and C and the span as $2r_1$. On this figure is shown, two wake trail-

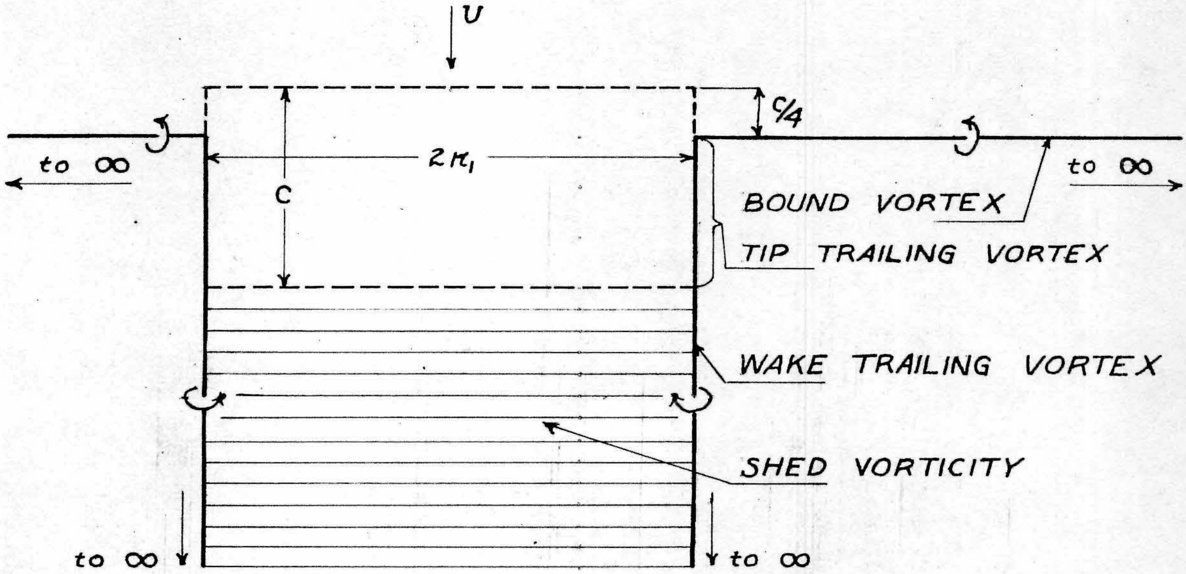

Fig. 1.5 - Vortices considered in this Problem

ing vortices, the shed vorticity, two bound and tip trailing vortices. The wake trailing, bound and tip vortices are labeled on one side only in figure 1.5.

In the problem which follows the effects of these various vortices are treated separately in the following sequence; (1) the wake trailing vortices, (2) the shed vorticity, (3) the bound and tip trailing vortices. It might be point out however, that this does not complete the problem, since the motion of the wing must be considered. This is taken up as the fourth item.

I-2 Vortex Pattern Equivalent to an Elliptical Wing

The choice of the proper vortex pattern corresponding to an elliptical wing involves essentially the choice of an equivalent chord and of an average distance between the trailing vortices.

In the subsequent treatment the evaluation of the forces shall be limited to the mid-span chord, and an elliptical spanwise distribution of the forces will be assumed.

The chord EF in figure 1.4 will therefore be taken the same as the maximum chord of the elliptical wing.

The average distance between the trailing vortices is derived from the following conditions. Consider the case of a stationary wing.

In the case of an elliptic lift distribution from the lifting line theory, a known value of the downwash is induced at the quarter chord. If the trailing vortices are concentrated as shown in the vortex pattern, their distance may be determined by the condition that they produce the same downwash at the quarter chord as if they were distributed according to an elliptic lift distribution.

This condition affords a method by which the span of the vortex pattern can be determined in order to correspond to the case of an elliptic wing.

The procedure is of course approximate and does not hold exactly for the oscillating wing. It is believed however that the approximation is satisfactory in the practical range of aspect ratios, frequencies, and velocities. From the lifting line theory, for the elliptical wing the downwash on the lifting line is given by the formula,

$$w = \frac{\Gamma_0}{2b}$$

where w = the downwash velocity,

Γ_0 = the circulation at midspan and

b = the wing span.

In case the wing of figure 1.4 is moving at constant airspeed and angle of attack (i.e., not oscillating), there is no shed vorticity and the trailing vortices - both tip and wake - are of constant strength throughout. The downwash at the point D of figure 1.4 can be computed from the Biot-Savart law and is

$$w = \frac{\Gamma_0}{2\pi r_1}$$

Here the strength of the trailing vortices of figure 1.4 is assumed to be the same as that of the elliptical wing at midspan. Equating the two expressions for the downwash, there is attained

$$\frac{\Gamma_0}{2b} = \frac{\Gamma_0}{2\pi r_1}$$

from which the span of an elliptical wing equivalent to the wing of Fig. 1.4 is given by

$$b = \pi r_1$$

The aspect ratio of any wing is defined as

$$AR = \frac{b^2}{S}$$

where the ligature AR symbolizes the aspect ratio, and S designates the wing plan form area. For an elliptical wing of span b and midchord length c, the area is given by

$$S = \frac{\pi bc}{4}$$

hence the aspect ratio is

$$AR = \frac{4b}{\pi c}$$

if the value $b = \pi r_1$ obtained previously is substituted into the equation for aspect ratio,

$$AR = \frac{4r_1}{c} \qquad (2.1)$$

This formula, when written as

$$r_1 = \frac{cAR}{4} \qquad (2.2)$$

gives a method by which the semispan of the hypothetical wing shown in figure 1.4 can be computed for a given elliptical wing.

I-3 Downwash Computation for the Wake Trailing Vortices

The next step is to determine the expressions for the downwash along the chord EF (figure 1.4, section I-1) due to the vortex pattern described in section I.1. This downwash cannot be computed explicitly since the circulation about the wing is at present unknown. For the purpose of carrying through these calculations the magnitude of the circulation about the wing will here be designated by the letter A (do not confuse with the A on the figures). The value of A will be determined in a later section. In this section the downwash due to the wake trailing vortices is computed, in section I-4 that due to the shed vorticity, and in section I-5 the downwash due to the tip trailing vortices is computed in combination with the bound vortices of figure 1.3, section I-1.

For the computation of the downwash due to the trailing vortices, figure 1.4, section I-1, has been redrawn; it is shown as figure 3.1. The upper part of this figure shows the plan view and the lower shows the elevation. In the plan view the two lines BC to infinity and $B'C'$ to infinity represent the wake trailing vortices. The wing is shown in phantom where the line BFB' is the trailing edge and EF is the chord. Also on figure 3.1 is shown the coordinate system. It is a double coordinate system with the point

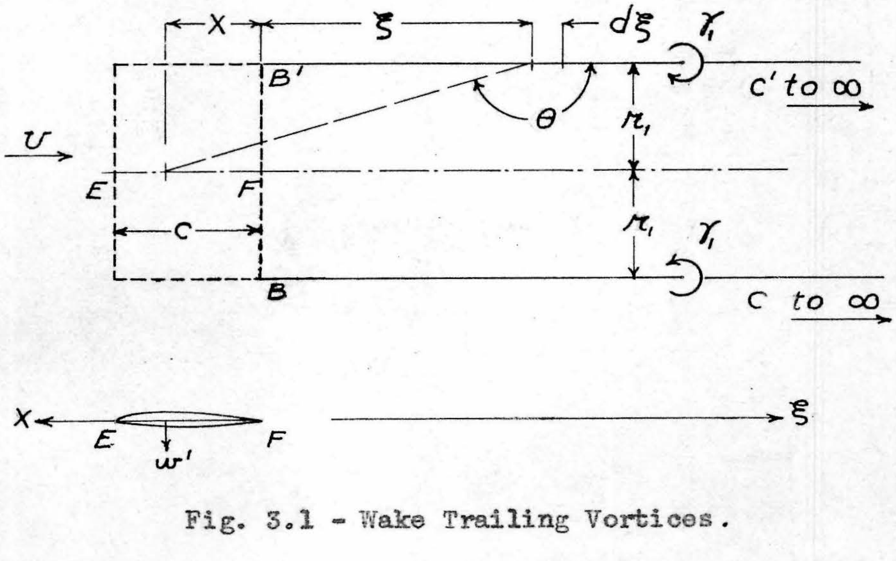

Fig. 3.1 - Wake Trailing Vortices.

F as the common origin. One coordinate system starts at the point F and extends positively to the right; it is designated by the Greek letter ξ. The other coordinate axis starts at F and extends positively to the left; it is designated by X. The two axes are clearly shown in the elevation of figure 3.1. Also in the elevation is shown the downwash vector w' which is taken as positive downward. On the lines BC to infinity, and $B'C'$ to infinity, are shown two curved arrows which represent the positive rotational sense of the vortex lines. Adjacent to each cruved arrow is the symbol γ_1 which represents the strength of the vortex line.

If a wing is oscillating, vortices of variable strength and rotational sense will be shed from the trailing edge. Since there must be continuity between the shed vorticity and the trailing vortices, it is evident that the trailing vortices will also be variable in character. The two trailing vortices are assumed to flow downstream with the relative wind and their strength γ_1 will vary with time and distance. If the oscillations of the wing are those of simple harmonic motion, the variation of γ_1 will be sinusoidal; hence

$$\gamma_1 = \bar{A}\, e^{i(\omega t - \lambda \xi)}$$

where $\bar{A}$ = a complex constant,

i = $\sqrt{-1}$,

ω = the angular frequency of the oscillating wing,

t = the time,

λ = 2π divided by the wave length (see appendix A) and

ξ = the coordinate of figure 3.1.

At any particular instant of time it is obvious that

$$\gamma_1 = A\, e^{-i\lambda\xi} \tag{3.1}$$

where A is the instantaneous value of the circulation about the trailing vortex at the trailing edge. Comparing (3.1) with the first expression it follows that

$$A = \bar{A} e^{i\omega t}$$

In the development which follows, the instantaneous values will be used; i.e., expression (3.1). As was mentioned in the first paragraph of this section, A is the circulation about the wing shown in figure 3.1; as defined above, it is the instantaneous value of that circulation. Between λ, ω, and U there exists a relationship. This relationship is set forth in section I-6 and is explained in appendix A. If a wing is oscillating at a given frequency and flying at a given air speed, the quantities λ, ω, and U are known.

To compute the downwash the Biot-Savart law is used. This law is given by Durand in volume I of "Aerodynamic Theory" page 137, see reference 1. As applied to the wake trailing vortices or figure 3.1, the Biot-Savart law is

$$dw' = 2\frac{\gamma_1 \, d\xi \sin\theta}{4\pi a^2}$$

where dw' is the element of downwash velocity, and the factor 2 is inserted in order that both vortices can be accounted for simultaneously.

From the geometry of figure 3.1 it is seen that

$$a = \sqrt{r_1^2 + (\xi + x)^2}$$

and

$$\sin\theta = \frac{r_1}{\sqrt{r_1^2 + (\xi + x)^2}}$$

Substituting equation (3.1) and the above, in the law of Biot-Savart, it becomes;

$$dw' = \frac{A r_1 e^{-i\lambda\xi} d\xi}{2\pi\left[r_1^2 + (\xi + x)^2\right]^{3/2}}$$

$$= \frac{A}{2\pi r_1} \frac{e^{-i\lambda\xi} d\xi}{\left[1 + \left(\frac{\xi + x}{r_1}\right)^2\right]^{3/2}}$$

In order to simplify the above expression the following approximation is made for the denominator of the above expression:

$$\frac{1}{\left[1 + \left(\frac{\xi + x}{r_1}\right)^2\right]^{3/2}} = \left[1 + 2\left(\frac{\xi + x}{r_1}\right)\right] e^{-2\left(\frac{\xi + x}{r_1}\right)}$$

In order to simplify the writing let $\frac{\xi + x}{r_1} = s$; then the expression becomes

$$f_1(s) = \frac{1}{\left[1 + s^2\right]^{3/2}} \tag{3.2}$$

and

$$f_2(s) = \left[1 + 2s\right] e^{-2s} \tag{3.3}$$

where $f_2(s)$ is the approximation to $f_1(s)$.

It will be noticed that both of these expressions are equal for $s = 0$ and $s = \infty$. The derivative of (3.2) is zero when $s = 0$ and since in equation (3.3) the same number occurs for the coefficient of s in the algebraic factor as occurs in the exponent of e , the derivative of (3.3) is zero when $s = 0$. This will be true for $f_2(s) = [1 + Ks]e^{-Ks}$ where K is any number > 0. The value of K is put equal to $K = 2$ in order that (3.3) satisfy another condition; the condition imposed is that the integral of (3.2) from zero to infinity equals the integral of (3.3) from zero to infinity. These integrals will be equal if K is taken as 2 in (3.3); i.e.,

$$\int_0^{\infty} \frac{ds}{[1+s^2]^{3/2}} = \int_0^{\infty} [1+2s]e^{-2s}\,ds$$

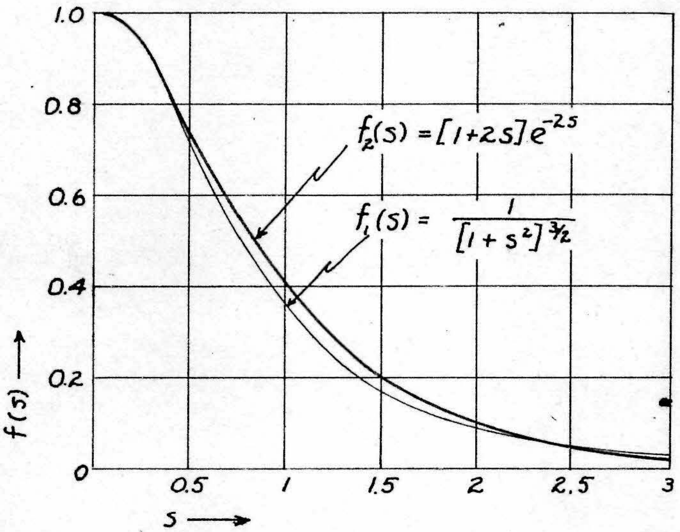

Fig. 3.2 - Graphs of Actual $f_1(s)$ and Approximating Function $f_2(s)$.

The graphs of the two functions are shown in figure 3.2 and from this it is apparent that the approximation is very satisfactory.

The expression for dw' can now be written as

$$dw' = \frac{A}{2\pi r_1^2}\left[1+2\left(\frac{\xi+x}{r_1}\right)\right]e^{-2\left(\frac{\xi+x}{r_1}\right)}e^{-i\lambda\xi}\,d\xi$$

or, integrating,

$$w' = \frac{A}{2\pi r_1^2}e^{-\frac{2x}{r_1}}\int_0^{\infty}\left[1+2\left(\frac{\xi+x}{r_1}\right)\right]e^{-\left(i\lambda+\frac{2}{r_1}\right)\xi}\,d\xi$$

This is easily integrated and after substituting the limits becomes

$$w' = \frac{Ae^{-\frac{2x}{r_1}}}{2\pi r_1^2}\left\{\frac{r_1+2x}{2+i\lambda r_1} + \frac{2}{r_1}\left[\frac{1}{\left(i\lambda+\frac{2}{r_1}\right)^2}\right]\right\}$$

This may be written in a more convenient form as

$$w' = \frac{A}{2\pi r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1}e^{-\frac{2x}{r_1}} + \frac{2x}{r_1}e^{-\frac{2x}{r_1}}\right\} \qquad (3.4)$$

This expression gives the downwash due to the trailing vortices alone, at any point x on the chord EF of figure 3.1.

I-4 Downwash Computation for the Shed Vorticity

In figure 1.4 the shed vortices are shown as individual lines parallel to the trailing edge of the wing, which connect the wake trailing vortices BC and $B'C'$. If the oscillatory motion of the wing is simple harmonic, or any continuously varying motion, the vortex lines will be shed so close together that they will lose their individual identity and form a continuous vortex sheet, or ribbon. Herein this vortex sheet or ribbon will be referred to as shed vorticity. As in the case of the wake trailing vortices, the shed vorticity is assumed to lie in a plane and flow down stream with the relative wind.

Figure 1.4 has again been redrawn and is shown as figure 4.1. The plan view is shown only, with the wing drawn in phantom; $B'FB$ and EF represent the trailing edge and chord respectively. The relative wind vector is also shown on this figure. The same coordinate system is used here as in the case of the wake trailing vortices. The lines which represent the shed vorticity on figure 1.4 are not shown in figure 4.1. However, there is shown on figure 4.1 an elementary portion of the shed vorticity having a span of $2r_1$, a width $d\xi$, which is at a distance ξ from the trailing edge. The curved arrow on the figure shows the rotational sense of the vortex element which is considered positive.

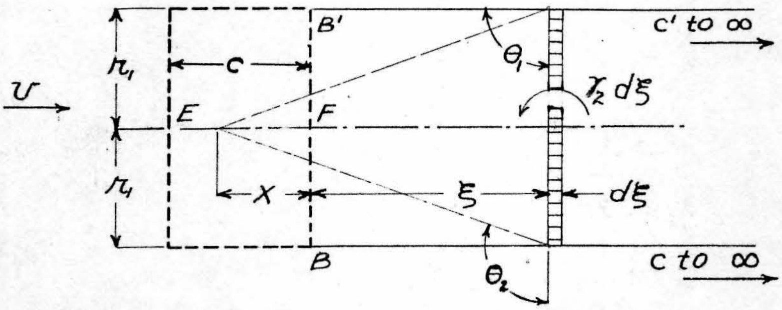

Fig. 4.1 - Shed Vorticity Element.

Adjacent to this curved arrow on figure 4.1 is the term $\gamma_2 d\xi$; this designates the strength of the element of shed vorticity. The symbol γ_2 is not the same kind of quantity as is the γ_1, used previously in connection with the wake trailing vortices. The difference is that γ_2 is a vortex density while γ_1 designates the strength of a wake trailing vortex at the point in question. Actually γ_2 is the vortex strength per unit length along the ξ axis. For sinusoidal variation of the shed vorticity, γ_2 can be written as

$$\gamma_2 = B e^{-i\lambda\xi} \tag{4.1}$$

where B (not to be confused with the B's which represent points on the figures) is the value of the vortex density when $\xi = 0$ - in other words, at the trailing edge. The value of B, as will be pointed out at the end of this section, has a direct relation to the A of equation (3.1) through the continuity existing between the shed and trailing vortices. The symbols i, λ and ξ have the same meaning as in section I-3. As was pointed out in that section the quantity λ is known, (See section I-6 and appendix A).

Since the element of shed vorticity has constant strength throughout its span, the element of downwash velocity at the point X on the chord EF is by the Biot-Savart law

$$dw'' = \frac{\gamma_2 d\xi}{4\pi(\xi + X)}(\cos\theta_1 - \cos\theta_2)$$

This downwash velocity which is due to the shed vorticity, is designated by the double prime accent marks. From the geometry of figure 4.1 it is evident that

$$\cos\theta_1 = \frac{r_1}{\sqrt{r_1^2 + (\xi + X)^2}}$$

and

$$\cos\theta_2 = \frac{-r_1}{\sqrt{r_1^2 + (\xi + X)^2}}$$

Substituting the value of γ_2 given by (4.1) and these expressions for the cosines, the element of downwash velocity becomes

$$dw'' = \frac{r_1 B e^{-i\lambda\xi} d\xi}{2\pi(\xi+X)\sqrt{r_1^2+(\xi+X)^2}}$$

Here again, in order to ease the integration, an approximation is made to eliminate the square root from the denominator of the above expression. As in section I-3 let

$$s = \frac{\xi + X}{r_1}$$

The expression for dw'' is rewritten as

$$dw'' = \frac{B e^{-i\lambda\xi} d\xi}{2\pi r_1 \left(\frac{\xi+X}{r_1}\right)\sqrt{1+\left(\frac{\xi+X}{r_1}\right)^2}} \qquad (4.2)$$

and the part of this expression to be approximated is

$$\frac{1}{\left(\frac{\xi+X}{r_1}\right)\sqrt{1+\left(\frac{\xi+X}{r_1}\right)^2}}$$

Substituting the new variable s and designating the new expression by $f_3(s)$, it becomes

$$f_3(s) = \frac{1}{s\sqrt{1+s^2}}$$

Here it will be noticed that this function tends to zero as s tends to infinity and that it tends to infinity as s tends to zero. It will also be noticed that when s is very small, say $s<0.1$, the radical $\sqrt{1+s^2}=1$ approximately; actually, with an error or less than one half percent. Hence as s tends to zero $f_3(s)$ tends to infinity as $\frac{1}{s}$. Therefore a function is desired which tends to infinity like $\frac{1}{S}$ as S tend to zero, and tends to zero as S tends to infinity. Let $f_4(S)$ designate this function which is to approximate $f_3(S)$; then $f_4(S)$ should become infinite like $\frac{1}{S}$ when S tends to zero. It is desirable to write

$$f_4(S) = \frac{1}{S} - \phi(S)$$

where $\phi(S)$ is a function which has a suitable value when $S = 0$ and tends to zero when S tends to infinity. It is also to be pointed out that $\phi(S)$ is to remain finite for $0 \leq S \leq \infty$. Writing the function as shown above has the advantage that the part which becomes infinite at $S = 0$ is separated from the part which remains finite over the range of S . Equating the functions $f_3(S)$ and $f_4(S)$ gives

$$\frac{1}{S\sqrt{1+S^2}} = \frac{1}{S} - \phi(S)$$

or

$$S\phi(S) = 1 - \frac{1}{\sqrt{1+S^2}}$$

The form $S\phi(S)$ is used rather than $\phi(S)$ because of the simplicity of the right hand side of the above equation. Since the exponential form integrates quite readily and combines easily with the $e^{-i\lambda\xi}$, already existent in the downwash formula (4.2), it is chosen for $\phi(S)$; thus

$$\phi(S) = K e^{-mS}$$

or

$$S\phi(S) = K S e^{-mS}$$

where constants K and m are positive real numbers.

Several values of K and m were arbitrarily taken and the resultant functions were plotted together with $1 - \frac{1}{\sqrt{1+S^2}}$. From these graphs it appeared that the best fit was obtained when $m = 1/6$ and $K = 0.3786$. The functions $0.3786\, S e^{-\frac{1}{6}S}$ and $1 - \frac{1}{\sqrt{1+S^2}}$ are shown in

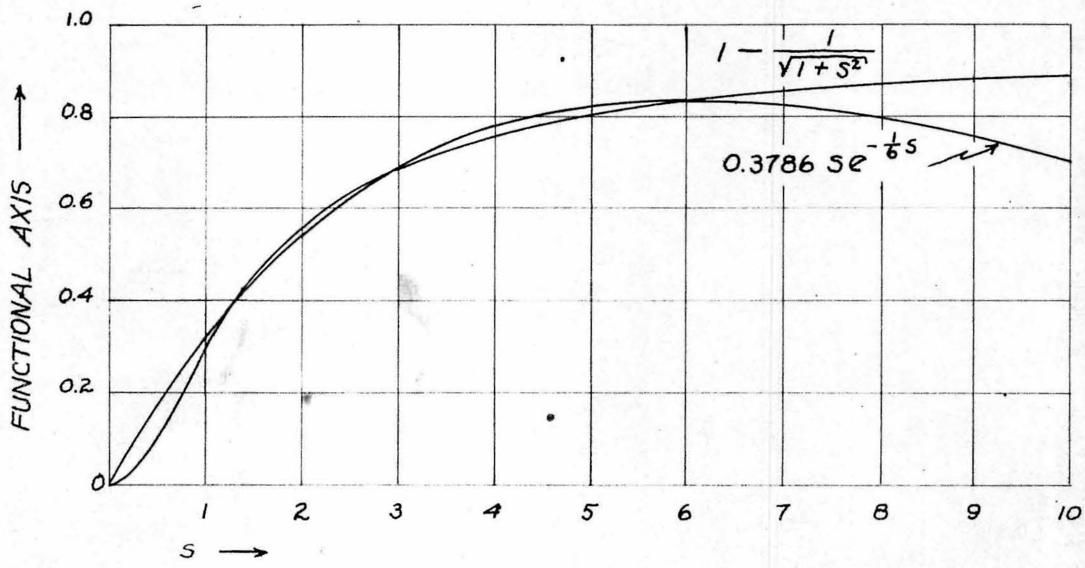

Fig. 4.2 - Graphs Showing Actual and Approximating Functions : $1 - \frac{1}{\sqrt{1+S^2}}$ and $0.3786\, S e^{-\frac{1}{6}S}$ Respectively.

figure 4.2. The number 0.3786 was obtained by making the two curves coincide at $S = 6$, which turns out to be the maximum of the approximating function. The calculation of this number is shown below. Thus, for $S = 6$

$$6Ke^{-1} = 1 - \frac{1}{\sqrt{1+6^2}}$$

or

$$K = \frac{e}{6}\left(1 - \frac{1}{\sqrt{37}}\right)$$

$$= 0.3786$$

It is to be noticed that this approximation fits well up to $s = 6$. Since the greatest contribution to the downwash is produced by small values of s, the deviation which appears for s greater than 6 is of minor importance - consequently $\phi(s)$ is chosen as

$$\phi(s) = 0.3786\, e^{-\frac{1}{6}s}$$

The function $f_4(s)$ now becomes

$$f_4(s) = \frac{1}{s} - 0.3786\, e^{-\frac{1}{6}s} \tag{4.3}$$

The function $f_3(s)$ is plotted in figures 4.3(a) and (b); figure 4.3(a) shows the curve plotted to a relatively large scale while figure 4.3(b) shows the functional scale reduced one twentieth.

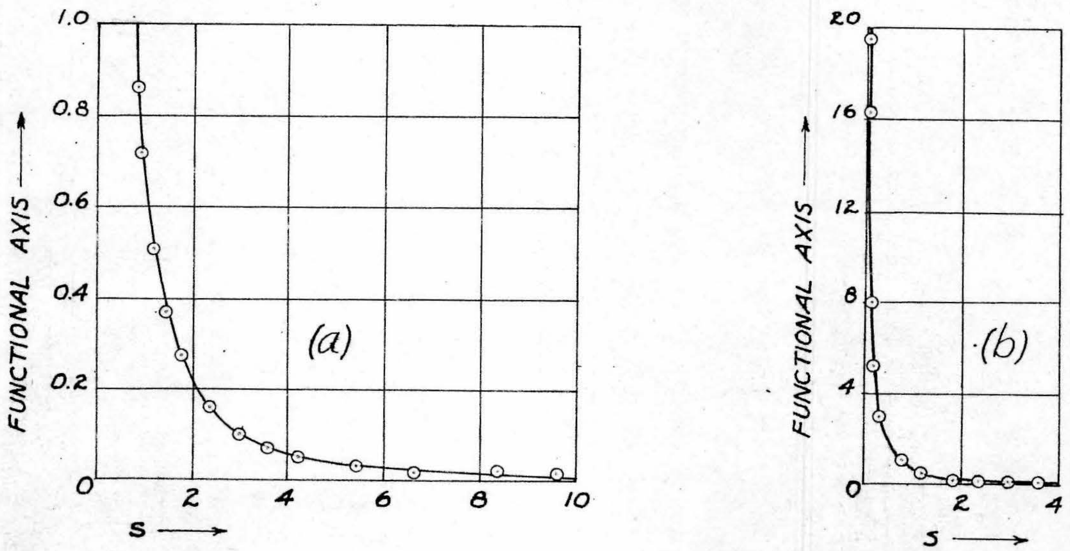

Fig. 4.3 - Graph showing $f_3(s) = \dfrac{1}{s\sqrt{1+s^2}}$,
Points computed from $f_4(s) = \dfrac{1}{s} - 0.3786\, e^{-\frac{1}{6}s}$.

The points shown on figure 4.3 are computed from equation (4.3) and plotted on both figures as small circles. It will be noticed that the approximation is most excellent for $s \leq 6$, and that as shown in figure 4.3(a) the deviation is only slight for $s > 6$. Substituting for s the quantity $\frac{\xi + x}{r_1}$ it is now possible to write with good approximation that

$$\frac{1}{\left(\frac{\xi+x}{r_1}\right)\sqrt{1+\left(\frac{\xi+x}{r_1}\right)^2}} = \frac{1}{\frac{\xi+x}{r_1}} - 0.3786\, e^{-\frac{1}{6}\left(\frac{\xi+x}{r_1}\right)}$$

Introducing this into equation (4.2) the element of downwash velocity can be written as

$$dw'' = \frac{B}{2\pi r_1}\left\{\frac{1}{\frac{\xi+x}{r_1}} - 0.3786\, e^{-\frac{1}{6}\left(\frac{\xi+x}{r_1}\right)}\right\} e^{-i\lambda\xi}\, d\xi$$

The downwash, for any point x along the chord due to the shed vorticity, is now given by the following integral:

$$w'' = \frac{B}{2\pi}\int_0^\infty \frac{e^{-i\lambda\xi}\, d\xi}{\xi + x} - \frac{0.3786\, B\, e^{-\frac{x}{6r_1}}}{2\pi r_1}\int_0^\infty e^{-\left(i\lambda + \frac{1}{6r_1}\right)\xi}\, d\xi$$

The first integral will be left in its indicated form, for it will be found later that in order to determine the circulation it must be integrated also with respect to x. As it turns out the mathematical procedure is simpler if the integration is carried out first with respect to x and secondly with respect to ξ. The second integral being elementary in nature can be evaluated at once. Thus,

$$\int_0^\infty e^{-\left(i\lambda + \frac{1}{6r_1}\right)\xi}\, d\xi = \left.\frac{e^{-\left(i\lambda + \frac{1}{6r_1}\right)\xi}}{-\left(i\lambda + \frac{1}{6r_1}\right)}\right]_0^\infty$$

$$= \frac{6r_1}{1 + 6i\lambda r_1}$$

Substituting this result in expression (4.4), it becomes

$$w'' = \frac{B}{2\pi}\int_0^\infty \frac{e^{-i\lambda\xi}\,d\xi}{\xi + x} - \frac{1.1358\,B\,e^{-\frac{x}{6h_1}}}{\pi(1+6i\lambda h_1)} \qquad (4.4)$$

It is to be pointed out that in expression (4.4) the first integral does not contain h_1 , hence it is independent of span or more specifically aspect ratio . This first integral is the one, which is obtained in the two dimensional theory while the second term which contains h_1 is consequently dependent on aspect ratio (see equation (2.1)). From this point of view the second term of (4.4) is a correction to the two-dimensional theory.

As previously stated there must be continuity between the trailing vortices and the shed vorticity. In figure 4.4, the upper horizontal line represents a segment of one of the wake trailing vortices of figure 4.1. The shaded part of width $d\xi$ represents a portion of the element of shed vorticity. This element is located at a distance ξ from the trailing edge of the wing. The curved arrows show the positive rotational sence of the wake trailing vortex and the element of shed vorticity. The strength of the trailing vortex at the point P_1 is taken as γ_1.

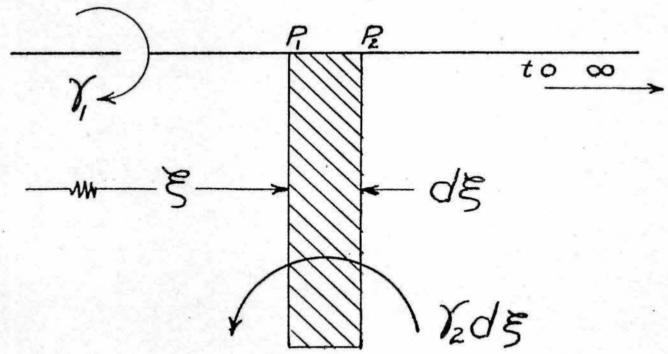

Fig. 4.4 - Showing Continuity of Trailing and Shed Vorticity.

Now since, the elementary distance P_1P_2 is in the direction of the positive ξ - axis, the strength of the trailing vortex at the point P_2 will be increased by the amount $d\gamma_1$; hence its total strength will be $\gamma_1 + d\gamma_1$. Therefore, by the principal of continuity of the vorticity, the difference between the wake trailing vortex strengths at P_1 and P_2 must be equal to the element of shed vorticity. Since the strength of the shed vortex element is $\gamma_2 \, d\xi$; it follows that

$$\gamma_1 - (\gamma_1 + d\gamma_1) = \gamma_2 \, d\xi$$

hence the expression for the continuity of the vorticity becomes

$$-d\gamma_1 = \gamma_2 \, d\xi$$

From expression (3.1), section I-3,

$$d\gamma_1 = -i\lambda A e^{-i\lambda\xi} d\xi$$

Substituting this for $d\gamma_1$, and expression (4.1) for γ_2, in the above continuity expression, it becomes

$$-(-i\lambda A e^{-i\lambda\xi} d\xi) = B e^{-i\lambda\xi} d\xi$$

from which it follows that

$$B = i\lambda A \tag{4.5}$$

The downwash equation (4.4) can now be written as

$$w'' = \frac{i\lambda A}{2\pi} \int_0^\infty \frac{e^{-i\lambda\xi} d\xi}{\xi + x} - \frac{1.1358 \, i\lambda A e^{-\frac{x}{6r_1}}}{\pi(1 + 6 i\lambda r_1)} \tag{4.6}$$

Here the expression is given in terms of the one unknown A which will be evaluated later.

I-5 Downwash due to Tip Trailing and Bound Vortices.

Since two-dimensional thin airfoil theory is to be applied to the chord EF figure 1.4, it is necessary to determine the effect of the downwash which is due to the bound vortex extending from point A to infinity on the left, and from point A' to infinity on the right as shown in figure 1.3. These bound vortices are in nature, a correction which reduces the wing of infinite aspect ratio, to that of finite aspect ratio.

In order to preserve continuity between the bound and the wake trailing vortices, the short vortex segments AB and $A'B'$ of figure 1.3 must be introduced. The effects of these vortex segments, herein called the tip trailing vortices, must also be considered in the downwash calculations. In this section the bound vortices are considered firstly, the tip trailing vortices secondly, and finally the two effects are combined in a simplifying approximation.

Figure 5.1 shows the hypothetical wing in phantom, where BFB' represents the trailing edge, and EF the chord. The two bound vortices are also shown, however the tip trailing vortex segments have been omitted. The positive rotational sense of the vortices is indicated by the curved arrows. The positive direction of the variable S, which measures distance along the vortex line, is shown in the upper part of figure 5.1. The positive direction of S is consistent with the right hand rule, used in electro magnetic theory.

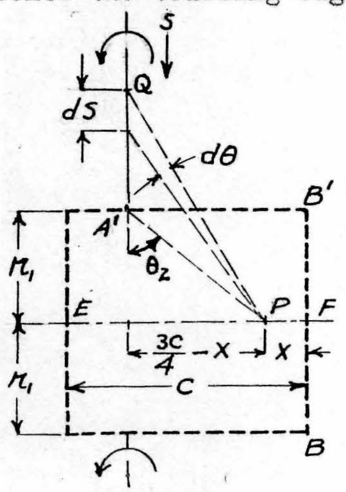

Fig. 5.1 - Bound Vortices

To compute the downwash at the point P figure 5.1, the Biot-Savart law is applied to the vortex element ds . This gives

$$dw_1 = -2\frac{A\sin\theta}{4\pi(PQ)^2}d\theta$$

Since the positive direction of the downwash is here taken as downward, and the bound vortex produces an upwash at the point P, the minus sign is introduced in the above expression. The factor two, preceding the right hand member, occurs because there are two of these bound vortices as shown in the figure 5.1. Here again the letter A designates the strength of the vortex. The acute angle, between the vortex and line PQ, is designated by θ. From the geometry of the figure, the distance $\overline{PQ}$ is

$$\overline{PQ} = \frac{\left(\frac{3c}{4} - x\right)}{\sin\theta}$$

The symbol c designates the length of chord EF. The element ds can be written as

$$ds = \frac{\overline{PQ}\,d\theta}{\sin\theta}$$

$$= \frac{\left(\frac{3c}{4} - x\right)d\theta}{\sin^2\theta}$$

Substituting the above for $\overline{PQ}$ and ds, in the expression for dw_1, it becomes

$$dw_1 = -\frac{A\sin\theta}{2\pi\left(\frac{3c}{4} - x\right)}d\theta$$

The limits of integration are taken as θ_1 and θ_2. The angle θ_2 is shown on figure 5.1, but θ_1 is not. In order to obtain θ_1 let the point Q recede to infinity; as it recedes the acute angle between PQ and the vortex line tends to zero. From this it follows that $\theta_1 = 0$, and hence

$$w_1 = -\frac{A}{2\pi\left(\frac{3c}{4}-x\right)}\int_0^{\theta_2}\sin\theta\, d\theta$$

$$= -\frac{A}{2\pi\left(\frac{3c}{4}-x\right)}\left[1-\cos\theta_2\right]$$

It is clear from figure 5.1 that

$$\cos\theta_2 = \frac{r_1}{\sqrt{r_1^2+\left(\frac{3c}{4}-x\right)^2}}$$

Substituting this in the expression for w_1, and rearranging algebraically, it becomes

$$w_1 = \frac{-A}{2\pi}\left[\frac{\left(\frac{3c}{4}-x\right)}{r_1^2+\left(\frac{3c}{4}-x\right)^2+r_1\sqrt{r_1^2+\left(\frac{3c}{4}-x\right)^2}}\right] \qquad (5.1)$$

In figure 5.2, only one half of the hypothetical wing is shown. In this figure EF represents the chord, and FB' is one half of the trailing edge. Only one tip trailing vortex is shown, and this in the upper part of the figure.

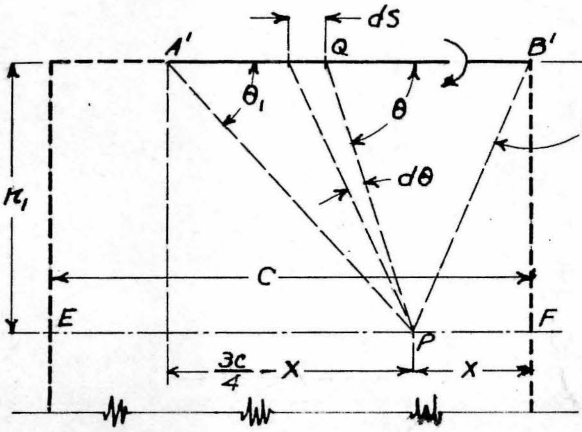

Fig. 5.2 - Tip Trailing Vortex

Since there are two tip trailing vortices, the element of downwash at the point P as given by the Biot-Savart law is

$$dw_2 = \frac{2A\sin\theta}{4\pi(PQ)^2}\,ds$$

Here the direction of the induced downwash velocity is downward, hence dw_2 is positive. From figure 5.2, the distance

$$\overline{PQ} = \frac{r_1}{\sin\theta}$$

and

$$ds = \frac{\overline{PQ}\,d\theta}{\sin\theta} = \frac{r_1\,d\theta}{\sin^2\theta}$$

Substituting the above in the expression for dw_2, it becomes

$$dw_2 = \frac{A \sin\theta}{2\pi r_1} d\theta$$

and

$$w_2 = \frac{A}{2\pi r_1}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta$$

$$= \frac{A}{2\pi r_1}\left[\cos\theta_1 - \cos\theta_2\right]$$

From figure 5.2, it is apparent that

$$\cos\theta_1 = \frac{\frac{3c}{4} - x}{\sqrt{r_1^2 + \left(\frac{3c}{4} - x\right)^2}}$$

and

$$\cos\theta_2 = \frac{-x}{\sqrt{r_1^2 + x^2}}$$

The final expression for w_2 now becomes

$$w_2 = \frac{A}{2\pi r_1}\left[\frac{\frac{3c}{4} - x}{\sqrt{r_1^2 + \left(\frac{3c}{4} - x^2\right)^2}} + \frac{x}{\sqrt{r_1^2 + x^2}}\right] \tag{5.2}$$

Let w''' designate the downwash due to the bound and tip trailing vortices, then

$$w''' = w_1 + w_2$$

Substituting for w_1 and w_2, expressions (5.1) and (5.2) but reversing the order of addition, w''' becomes

$$w''' = \frac{A}{2\pi}\left[\frac{\frac{3c}{4} - x}{r_1\sqrt{r_1^2 + \left(\frac{3c}{4} - x\right)^2}} + \frac{x}{r_1\sqrt{r_1^2 + x^2}} - \frac{\frac{3c}{4} - x}{r_1^2 + \left(\frac{3c}{4} - x\right)^2 + r_1\sqrt{r_1^2 + \left(\frac{3c}{4} - x\right)^2}}\right] \tag{5.3}$$

At this point it is very advisable to substitute for r_1, its equivalent in terms of aspect ratio, i.e. expression (2.2) section I-2. Performing this operation on expression (5.3), and simplifying algebraically, gives

$$w''' = \frac{A}{2\pi c}\left[\frac{\frac{16}{AR^2}\left(\frac{3}{4}-\frac{x}{c}\right)}{\sqrt{1+\frac{16}{AR^2}\left(\frac{3}{4}-\frac{x}{c}\right)^2}} + \frac{\frac{16}{AR^2}\frac{x}{c}}{\sqrt{1+\frac{16}{AR^2}\left(\frac{x}{c}\right)^2}}\right.$$

$$\left. - \frac{\frac{16}{AR^2}\left(\frac{3}{4}-\frac{x}{c}\right)}{1+\frac{16}{AR^2}\left(\frac{3}{4}-\frac{x}{c}\right)^2+\sqrt{1+\frac{16}{AR^2}\left(\frac{3}{4}-\frac{x}{c}\right)^2}}\right] \qquad (5.4)$$

Equation (5.4) is approximated by the following parabolic expression:

$$w''' = \frac{A}{2\pi c}\left[a_0 + a_1\left(\frac{x}{c}\right) + a_2\left(\frac{x}{c}\right)^2\right] \qquad (5.5)$$

Expressions (5.4) and (5.5) are made to agree at the trailing edge $\frac{x}{c}=0$, at the mid-point of chord EF, $\frac{x}{c}=\frac{1}{2}$, and at the leading edge $\frac{x}{c}=1$. By this means, the quantities a_0 , a_1, and a_2 are evaluated as functions of the aspect ratio. Thus for $\frac{x}{c}=0$ expression (5.4) becomes

$$w''' = \frac{A}{2\pi c}\left[\frac{\frac{12}{AR^2}}{\sqrt{1+\frac{9}{AR^2}}} - \frac{\frac{12}{AR^2}}{1+\frac{9}{AR^2}+\sqrt{1+\frac{9}{AR^2}}}\right]$$

and expression (5.5) becomes

$$w''' = \frac{A}{2\pi c}\left[a_0\right]$$

Equating the above two expressions gives

$$a_0 = \frac{\frac{12}{AR^2}}{\sqrt{1+\frac{9}{AR^2}}} - \frac{\frac{12}{AR^2}}{1+\frac{9}{AR^2}+\sqrt{1+\frac{9}{AR^2}}}$$

and simplifying the above algebraically, a_0 becomes

$$a_0 = \frac{4}{3\mathit{AR}}\sqrt{\mathit{AR}^2+9} - \frac{4}{3} \tag{5.6}$$

Using the value $\frac{x}{c} = \frac{1}{2}$ in the same way gives

$$a_0 + \frac{a_1}{2} + \frac{a_2}{4} = \frac{\frac{4}{\mathit{AR}^2}}{\sqrt{1+\frac{1}{\mathit{AR}^2}}} + \frac{\frac{8}{\mathit{AR}^2}}{\sqrt{1+\frac{4}{\mathit{AR}^2}}} - \frac{1}{1+\frac{1}{\mathit{AR}^2}+\sqrt{1+\frac{1}{\mathit{AR}^2}}}$$

and the result for $\frac{x}{c} = 1$ is

$$a_0 + a_1 + a_2 = \frac{-\frac{4}{\mathit{AR}^2}}{\sqrt{1+\frac{1}{\mathit{AR}^2}}} + \frac{\frac{16}{\mathit{AR}^2}}{\sqrt{1+\frac{16}{\mathit{AR}^2}}} + \frac{\frac{4}{\mathit{AR}^2}}{1+\frac{1}{\mathit{AR}^2}+\sqrt{1+\frac{1}{\mathit{AR}^2}}}$$

Since a_0 is known, the above two equations can be solved simultaneously for a_1 and a_2. The results of this solution are given below:

$$a_1 = \frac{20}{\mathit{AR}}\sqrt{\mathit{AR}^2+1} + \frac{32\sqrt{\mathit{AR}^2+4}}{\mathit{AR}(\mathit{AR}^2+4)} - \frac{4}{\mathit{AR}}\sqrt{\mathit{AR}^2+9} - \frac{16\sqrt{\mathit{AR}^2+16}}{\mathit{AR}(\mathit{AR}^2+16)} - 16 \tag{5.7}$$

and

$$a_2 = -\frac{24}{\mathit{AR}}\sqrt{\mathit{AR}^2+1} - \frac{32\sqrt{\mathit{AR}^2+4}}{\mathit{AR}(\mathit{AR}^2+4)} + \frac{8}{3\mathit{AR}}\sqrt{\mathit{AR}^2+9} + \frac{32\sqrt{\mathit{AR}^2+16}}{\mathit{AR}(\mathit{AR}^2+16)} + \frac{64}{3} \tag{5.8}$$

To test the fit of the above approximation, curves are shown in figure 5.4 for both the actual and the approximate formula. The cases considered are for aspect ratios equal to $1, \sqrt{2}, 2,$ and 3. The greatest deviation occurs for aspect ratio one. For aspect ratio $\sqrt{2}$, the deviation is small and for 2 or greater, the deviation is practically negligible. Figure 5.5 shows the variation of a_0, a_1 and a_2, with aspect ratio. In table 1 the values of a_0, a_1 and a_2 are given for aspect ratios from 1 to 12 inclusive.

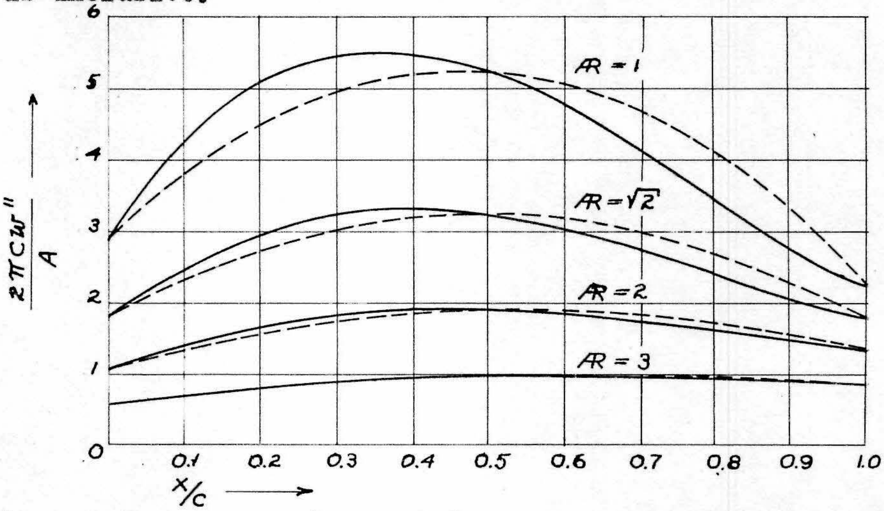

Fig. 5.4 - Curves of Actual (———) and Approximating (-----) Expressions for Tip Trailing and Bound Vortices Combined.

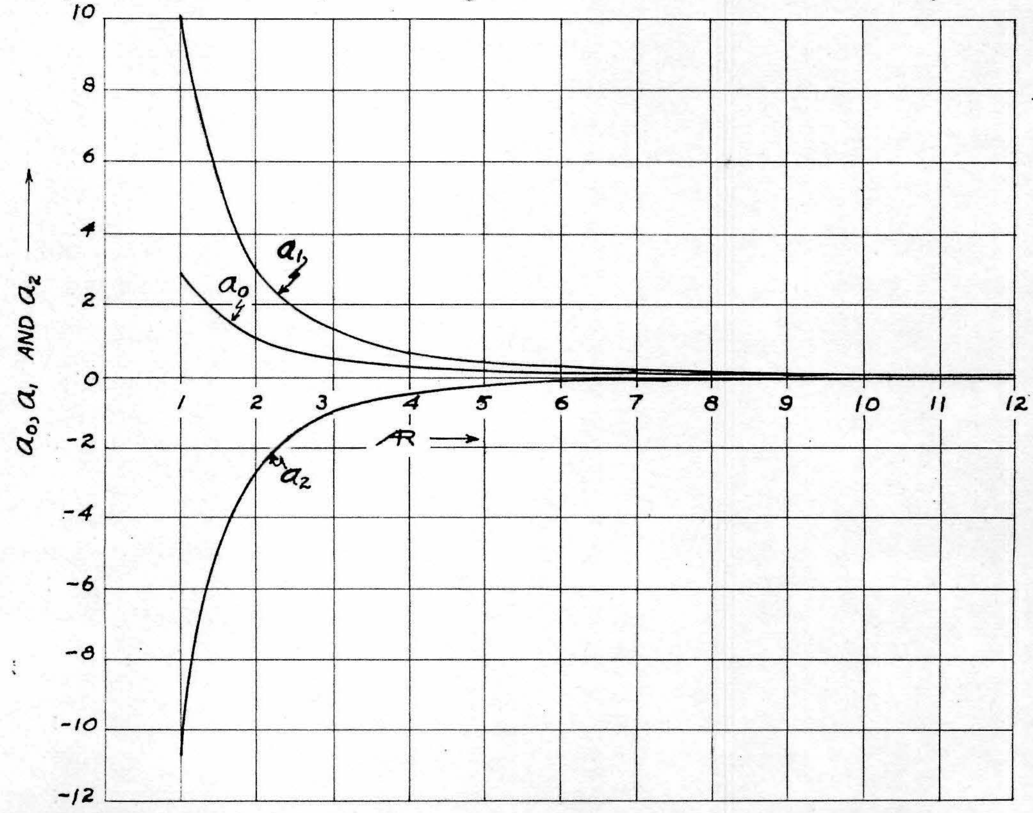

Fig. 5.5 - Curves of a_0, a_1, a_2 plotted against Aspect Ratio

TABLE 1

Values of a_0, a_1, and a_2 used in expression (5.5),

section I-5; i.e., $w''' = \frac{A}{2\pi c}\left[a_0 + a_1\left(\frac{x}{c}\right) + a_2\left(\frac{x}{c}\right)^2\right]$

AR	a_0	a_1	a_2
1	2.883037	10.065423	-10.724751
$\sqrt{2}$	1.7936107	5.685005	- 5.710929
1.5	1.6480907	5.129194	- 5.087805
2	1.0703667	3.017580	- 2.771230
3	.5522847	1.3167300	- 1.0187187
4	.3333334	.6972677	- .4466041
5	.2215873	.4200122	- .2210550
6	.15737865	.2772132	- .1199667
7	.1172900	.19556103	- .06999877
8	.09066729	.1451070	- .0432964
9	.07212334	.11185612	- .0280848
10	.05870753	.0888582	- .0189596
11	.04869749	.0723089	- .0132311
12	.04103521	.0600082	- .0094990

I-6 Wing oscillation replaced by equivalent downwash

In this section, the downwash velocity due to the motion of the wing is set forth. The motion of an oscillating wing can be separated into two categories. The first is the wing's forward velocity of flight which here is assumed as a constant horizontal velocity of magnitude U, and the second is the vibratory motion of the wing in which the displacements are assumed to be small.

Since motion is relative it can be analysed by considering the wing, at rest, and the air in motion. From this point of view, the air will stream over the wing with the velocity U, while at the same time there will be perturbation velocities, oscillatory in character, acting about the wing. This perturbation velocity is due to the presence of the wing, and from the fact that a condition of oscillation has been imposed. If the wing is taken as a thin flat plate, its thickness alone will cause no disturbance to the airflow. If the thin flat plate is set at zero stationary angle of attack, this cause for airflow disturbance is eliminated. With these two conditions present, the perturbation velocity about the wing is due only to the imposed condition of flutter. Since the theory developed herein is linear, there is no loss of generality by the stipulations of the thin flat plate, and the zero stationary angle of attack. The effects of thickness, and the stationary angle of attack can be simply added to the final result by the process of superposition.

The perturbation velocity about the wing can be thought of as arising from two sources. The first is that which is induced by the vortex pattern exterior to the wing, and the second is that which is due to the vibratory motion of the wing. The vertical component of this perturbation velocity is the downwash. In sections I-3, I-4, and I-5 the downwash velocities due to vortex pattern exterior to the wing were determined. In this section the downwash due to the oscillating motion of the wing is determined.

In the kinematics of the rigid body, it is often convenient to describe the velocity by specifying the linear velocity of some point in the body, and the angular velocity about an axis through that point. The oscillating wing conveniently lends itself to this description. The point here considered is the mid-point of the chord. The motion of this point turns out to be a rectilinear motion which is perpendicular to the horizontal velocity U, and the span. With this is associated the angular velocity about an axis through the mid-point of the chord. The axis referred to is taken parallel to the span. Rotational components about the chord are not considered.

The rectilinear motion of the wing which is perpendicular to the velocity vector U and the span, is called in flutter theory, the translatory oscillations. The angular motion, about an axis through the mid-point of the chord and parallel to the span, is called rotational oscillations.

Consider first, the translatory oscillations. Figure 6.1 shows the chord EF, and the velocity vector U which represents the undisturbed air stream. The wing is assumed to be vibrating up and down with simple harmonic motion, in such a manner that the chord always remains parallel to itself.

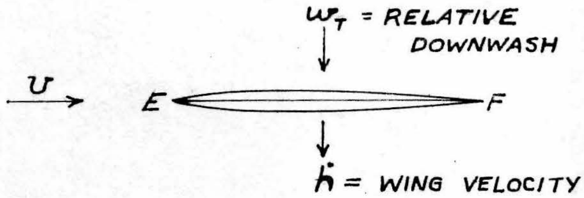

Fig. 6.1 - Translatory Oscillations

In figure 6.1, $\dot{h}$ represents the instantaneous velocity of the wing and is considered positive when downward. This notation is in agreement with Theodore Theodorsen's, "General Theory of Aerodynamic Instability and the Mechanism of

Flutter", N.A.C.A. Technical Report No. 496 (reference 4). It is evident, that when the wing is moving downward, the velocity of the relative wind due to this motion is upward, hence the downwash

$$w_T = -\dot{h} \tag{6.1}$$

It should be observed, that when the downwash w_T is positive, the lift is downward or negative.

Since $\dot{h}$ was given as the instantaneous value of the wing motion, consequently w_T is the instantaneous value of the downwash. It is true, however, that the downwash w_T is the same for all points of the chord EF. As simple hamonic motion is assumed the instantaneous value of the downwash can be written as

$$w_T = \overline{w_T}\, e^{i\omega t} \tag{6.2}$$

where $\overline{w_T}$ is the complex amplitude of the downwash velocity, and ω is the angular frequency.

Here again, the complex quantity $e^{i\omega t}$ is used owing to the convenience it affords in the mathemathical operations. As is shown in appendex A, if T designates the period in seconds, then

$$T = \frac{2\pi}{\omega}$$

The frequency f in cycles per second is the reciprocal of the period T, hence

$$f = \frac{\omega}{2\pi}$$

The wave length, as is given in section I-3 is $\frac{2\pi}{\lambda}$ and since the wave length is equal to the velocity U, divided by the frequency f, it can be written that

$$\frac{2\pi}{\lambda} = \frac{U}{\frac{\omega}{2\pi}}$$

or

$$\omega = \lambda U \qquad (6.3)$$

A detailed explanation of the above is given in appendix A.

In the case of the rotational oscillations, the undisturbed velocity U is considered as constant. The wing is assumed to oscillate about an axis through the midpoint of the chord and parallel to the span. Figure 6.2 shows the wing which is assumed as a thin flat plate. The angle α, is taken as positive when the wing produces an upward, or positive lift. The angular velocity $\dot{\alpha} = \frac{d\alpha}{dt}$ is positive with increasing α, which is shown as clockwise on the figure. The line EF represents the wing chord. The midpoint of the chord is the origin O for the variable X, positive to the right.

Consider first the effect of the angle α, assuming at present that $\dot{\alpha} = 0$. Resolving the undisturbed air stream U into components along and normal to the wing chord EF, it will be found that the normal component is $U \sin \alpha$.

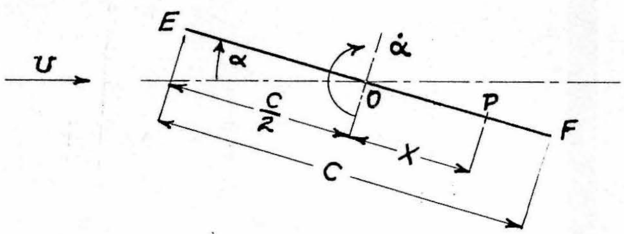

Fig. 6.2 - Rotational Oscillations

This same condition can be assimilated if the wing is considered to be at zero angle of attack and in a downward motion, with a speed $U \sin \alpha$. In this case the air would have an upward motion of $U \sin \alpha$. Since this is considered as negative in terms of downwash, hence the downwash caused by angle of attack is $-U \sin \alpha$. In the case of wing flutter, the angle α is no small that the sine may be replaced by its angle in radians, hence the downwash caused by angle of attack alone can be given as $-U\alpha$.

The above must be combined with the downwash due to the angular velocity $\dot{\alpha}$. The velocity of the point P, on the chord EF shown in figure

6.2, is equal to $\dot{\alpha} x$. This velocity is also normal to the chord. The downwash of the relative wind, due to this cause, is apparently $-\dot{\alpha} x$. Combining this value with that due to angle of attack, gives

$$w_R = -U\alpha - \dot{\alpha} x \tag{6.4}$$

where w_R is the downwash velocity due to rotational oscillations.

The usual procedure is to replace the wing by the downwash given in expression (6.4). This expression is valid only for $-\frac{c}{2} \leq x \leq \frac{c}{2}$. Attention is called to the fact that the origin, in this section, is taken at the midpoint of the chord.

To point out the oscillatory character of the angle α, and to put equation (6.4) in its final form, α is taken as

$$\alpha = \bar{\alpha} e^{i\omega t}$$

where $\bar{\alpha}$ is the amplitude. From this the angular velocity $\dot{\alpha}$ becomes

$$\dot{\alpha} = \bar{\alpha} i\omega e^{i\omega t}$$

Substituting the above in expression (6.4), it becomes

$$w_R = -U\bar{\alpha} e^{i\omega t} - \bar{\alpha} i\omega x e^{i\omega t}$$

$$= -(U + i\omega x)\bar{\alpha} e^{i\omega t}$$

If α is substituted for $\bar{\alpha} e^{i\omega t}$, the expression takes the form

$$w_R = -(U + i\omega x)\alpha \tag{6.5}$$

If expression (6.1) is added to (6.5) the result is

$$w_o = -\dot{h} - U\alpha - i\omega\alpha x \tag{6.6}$$

Here w_o represents the equivalent downwash due to translatory and rotational oscillations of the wing. Expression (6.6) is the form which will be used in computing the circulation which is carried out in Chapter II. It should be noticed, that in the above expression, $\dot{h}$ and α are instantaneous values, hence w_0 is the instantaneous value of the downwash.

Chapter II

CIRCULATION

II-1 Munk's Integral

In chapter I, the vortex pattern was introduced and explained. In addition the downwash due to this vortex pattern was determined. In the case of the wake trailing, the shed, the tip trailing and the bound vorticity, there appeared an unknown quantity A, which represents the circulation about the wing. It is the problem of chapter II, to determine this quantity. This is accomplished by an application of Munk's integral to the chord EF of the hypothetical wing shown in figure 1.4 section I-1.

Munk's integral is given by von Kármán and Burgers, as expression (9.15) on page 46, volume II of "Aerodynamic Theory", (see reference 2). The form in which it is given corresponds to the case of a stationary thin air-foil with an arbitrary camber line $y = f(x)$. It is written

$$\Gamma = -4aV\int_0^{\pi} \frac{dy}{dx}(1+\cos\tau)\,d\tau \qquad (1.1)$$

where Γ is the measure of the circulation about the wing. The letter V, appearing in this equation, designates the velocity of the undisturbed air, and is the same as the symbol U, used herein. The variable x, measures the distance along the wing chord and is related to τ, in that

$$x = 2a\cos\tau$$

In figure 1.1, the x-axis is shown along with the y-axis and the arc which represents the thin airfoil. The above relation between x and τ is obtained from the conformal transformation of a circle into an airfoil, in which a is the radius of the circle. The factor 2 is due

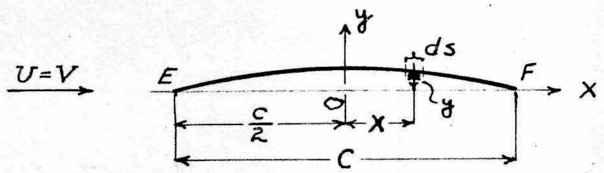

Fig. 1.1 -
Coordinate Axes and Airfoil

to the equation of transformation. When $\tau = 0$ the variable $x = 2a$ and this corresponds to the point F which designates the trailing edge shown in figure 1.1. The origin O of this figure is at the midpoint of the chord. The distance $OF = 2a$ is then one half a chord length which is designated here, by the letter c ; hence

$$a = \frac{c}{4}$$

The variable y, is the ordinate of the thin airfoil at the distance x from the origin O, as is shown in the figure. The derivative $\frac{dy}{dx}$ gives the slope of the wing section at the point (x, y).

Munk's integral in the form (1.1) may easily be transformed to the case of a symmetrical airfoil, lying along the x-axis, and submitted to an arbitrary downwash distribution along the chord. To this effect the slope can be considered as the angle of attack, relative to the undisturbed air, for a small element ds of airfoil arc located at the point (x, y), as shown in figure 1.1. It was shown in section I-6, that an angle of attack is equivalent to a downwash. Here, the angle of attack varies along the airfoil arc from leading edge to trailing edge. This angle is now to be replaced by a properly distributed downwash along the chord EF. From this it follows, that the distributed downwash equivalent to the airfoil is

$$w = V\frac{dy}{dx}$$

or in the notation used herein,

$$w = U\frac{dy}{dx}$$

Substituting the above results in equation (1.1), it becomes

$$\Gamma = -c\int_0^{\pi} w(1+\cos\tau)\,d\tau \qquad (1.2)$$

Since $w = w' + w'' + w''' + w_0$, expression (1.2) can be separated into four integrals and each integral evaluated separately.

The relation between x and τ can now be written as

$$x = \frac{c}{2}\cos\tau \qquad (1.3)$$

In the above discussion, the origin for x was taken as the mid-point of the chord. With the exception of w_0, section I-6, the origin in chapter I was taken at the trailing edge. It is therefore necessary to translate the origin to the trailing edge. The only expression affected, in the above formulae, is the relation (1.3). Figure 1.2 shows the chord EF, and its midpoint O, the point E being the leading edge, and F the trailing edge. For the purpose of distinction let the abscissa OP equal x', then equation (1.3) becomes

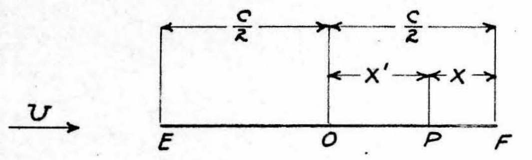

Fig. 1.2 - Translation of Axes

$$x' = \frac{c}{2}\cos\tau$$

From the figure it is apparent that

$$x' + x = \frac{c}{2}$$

hence

$$x' = \frac{c}{2} - x$$

Substituting this in the above relation, it becomes

$$\frac{c}{2} - x = \frac{c}{2}\cos\tau$$

or

$$x = \frac{c}{2}(1 - \cos\tau) \qquad (1.4)$$

For this expression, the origin is now at the trailing edge and the positive direction of x is toward the leading edge.

II-2 Circulation due to Wake Trailing Vortices

Expression (3.4) section I-3 gives the downwash due to the wake trailing vortices alone. Substituting relation (1.4) in this expression, it becomes

$$w' = \frac{A}{2\pi r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1}e^{-\frac{c}{r_1}(1-\cos\tau)} + \frac{c}{r_1}(1-\cos\tau)\,e^{-\frac{c}{r_1}(1-\cos\tau)}\right\}$$

$$= \frac{A\,e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1}e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left[e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right]\right\}$$

Let Γ' designate the circulation about the wing due to the wake trailing vortices alone. Introducing w' in Munk's integral, (1.2) of section II-1, it becomes

$$\Gamma' = \frac{-Ac\,e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1}\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}(1+\cos\tau)\,d\tau + \frac{c}{r_1}\left[\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}(1+\cos\tau)\,d\tau - \int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos\tau(1+\cos\tau)\,d\tau\right]\right\}$$

This expression involves three types of integrals. It is possible to express these integrals in terms of Bessel functions and this derivation is given in appendix B. These three types are given below together with their results:

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\,d\tau = \pi\,I_0\left(\tfrac{c}{r_1}\right) \tag{2.1}$$

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos\tau\,d\tau = \pi\,I_1\left(\tfrac{c}{r_1}\right) \tag{2.2}$$

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos 2\tau\,d\tau = \pi\,I_2\left(\tfrac{c}{r_1}\right) \tag{2.3}$$

where $I_0\left(\frac{c}{r_1}\right)$, $I_1\left(\frac{c}{r_1}\right)$, and $I_2\left(\frac{c}{r_1}\right)$ are the so-called modified Bessel functions of orders zero, one, and two. There is derived in appendix B, a recurrence formula between these three functions which is

$$I_2\left(\tfrac{c}{r_1}\right) = I_0\left(\tfrac{c}{r_1}\right) - \frac{2r_1}{c}\,I_1\left(\tfrac{c}{r_1}\right)$$

By means of this formula integral (2.3) can be expressed as

$$\int_0^\pi e^{\frac{c}{r_1}\cos\tau}\cos 2\tau\, d\tau = \pi\left[I_0\left(\tfrac{c}{r_1}\right) - \tfrac{2r_1}{c} I_1\left(\tfrac{c}{r_1}\right)\right] \qquad (2.4)$$

This greatly simplifies the numerical computations since it is now only necessary to evaluate $I_0(\frac{c}{r_1})$ and $I_1(\frac{c}{r_1})$.

Series expansions for these two functions are given in appendix B. There are tables which give the modified Bessel functions as high as the eleventh order, however, for orders greater than zero and one, they are much abridged. The values used here were either computed by the series expansions, or taken from "Mathematical Tables", Volume VI published by the British Association for the Advancement of Science (Cambridge 1937). These tables are most extensive, but give only $I_0(\frac{c}{r_1})$ and $I_1(\frac{c}{r_1})$. Consequently, when the modified Bessel functions occur in the formulae developed herein, they are reduced so as only to contain modified Bessel functions of order zero and one.

Substituting in the expression for Γ', the results of integrals (2.1), (2.2), and (2.4), eliminating r_1 by means of expression (2.2) of section I-2, and rearranging algebraically, the expression takes the form

$$\Gamma' = \frac{-A\, e^{-\frac{4}{AR}}}{AR\left(1+\frac{i\lambda c AR}{8}\right)}\left\{\frac{1}{1+\frac{i\lambda c AR}{8}}\left[I_0\left(\tfrac{4}{AR}\right) + I_1\left(\tfrac{4}{AR}\right)\right] + I_0\left(\tfrac{4}{AR}\right) + 2I_1\left(\tfrac{4}{AR}\right)\right\} \qquad (2.5)$$

It is very convenient to symbolize the above expression, hence a symbol F_T is defined such that

$$F_T = \frac{e^{-\frac{4}{AR}}}{AR\left(1+\frac{i\lambda c AR}{8}\right)}\left\{\frac{1}{1+\frac{i\lambda c AR}{8}}\left[I_0\left(\tfrac{4}{AR}\right) + I_1\left(\tfrac{4}{AR}\right)\right] + I_0\left(\tfrac{4}{AR}\right) + 2I_1\left(\tfrac{4}{AR}\right)\right\} \qquad (2.6)$$

It will be noticed that F_T is a function of both λ and AR. Using F_T the expression for Γ' can now be written as

$$\Gamma' = -A\, F_T \qquad (2.7)$$

II-3 Circulation due to Shed Vorticity

The downwash for the shed vorticity is given by expression (4.6) section I-4. Substituting for x its equivalent as given by expression (1.4) of section II-1 the downwash formula can be written

$$w'' = \frac{i\lambda A}{\pi c}\int_0^\infty \frac{e^{-i\lambda\xi}\,d\xi}{\frac{2\xi}{c}+1-\cos\tau} - \frac{1.1358\, i\lambda A}{\pi(1+6i\lambda h_1)} e^{-\frac{c}{12h_1}(1-\cos\tau)}$$

Substituting this for w in Munk's integral, expression (1.2) of section II-1, the circulation takes the form

$$\Gamma'' = -\frac{i\lambda A}{\pi}\int_0^\pi\int_0^\infty \frac{e^{-i\lambda\xi}(1+\cos\tau)}{\frac{2\xi}{c}+1-\cos\tau}\,d\xi\,d\tau$$

$$+\frac{1.1358\, i\lambda A}{\pi(1+6i\lambda h_1)} e^{-\frac{c}{12h_1}}\int_0^\pi e^{\frac{c}{12h_1}\cos\tau}(1+\cos\tau)\,d\tau$$

where Γ'' is the circulation due to the shed vorticity alone.

The second of the above two integrals being simpler will be treated first; thus

$$\int_0^\pi e^{\frac{c}{12h_1}\cos\tau}(1+\cos\tau)\,d\tau = \int_0^\pi e^{\frac{c}{12h_1}\cos\tau}\,d\tau + \int_0^\pi e^{\frac{c}{12h_1}\cos\tau}\cos\tau\,d\tau$$

This integration follows immediately from formulae (B8) and (B13) of appendix B, if $\frac{c}{h_1}$ is replaced by $\frac{c}{12h_1}$ and becomes

$$\int_0^\pi e^{\frac{c}{12h_1}\cos\tau}(1+\cos\tau)\,d\tau = \pi I_0\left(\tfrac{c}{12h_1}\right) + \pi I_1\left(\tfrac{c}{12h_1}\right)$$

where $I_0\left(\frac{c}{12h_1}\right)$ and $I_1\left(\frac{c}{12h_1}\right)$ are the modified Bessel functions of order zero and one. The expression for the circulation can now be written as

$$\Gamma'' = -\frac{i\lambda A}{\pi}\int_0^\pi\int_0^\infty \frac{e^{-i\lambda\xi}(1+\cos\tau)}{\frac{2\xi}{c}+1-\cos\tau}\,d\xi\,d\tau$$

$$+\frac{1.1358\, i\lambda c A}{1+6i\lambda h_1} e^{-\frac{c}{12h_1}}\left[I_0\left(\frac{c}{12h_1}\right)+I_1\left(\frac{c}{12h_1}\right)\right]$$

As was done with expression (2.6) in section II-2, it is convenient to introduce a symbol to represent the latter part of the above formula. The symbol used is F_S and for reasons which will appear later it is introduced with a minus sign; thus

$$F_s = -\frac{1.1358\, i\lambda c}{1 + 6 i\lambda r_1} e^{-\frac{c}{12 r_1}} \left[I_0\left(\frac{c}{12 r_1}\right) + I_1\left(\frac{c}{12 r_1}\right) \right]$$

If r_1 is eliminated by expression (2.2) section I-2, F_s takes the following form

$$F_s = -\frac{1.1358\, i\lambda c}{1 + \frac{3 i\lambda c \mathcal{R}}{2}} e^{-\frac{1}{3\mathcal{R}}} \left[I_0\left(\frac{1}{3\mathcal{R}}\right) + I_1\left(\frac{1}{3\mathcal{R}}\right) \right] \tag{3.1}$$

The expression for Γ'' can now be written as

$$\Gamma'' = -\frac{i\lambda A}{\pi} \int_0^{\pi}\int_0^{\infty} \frac{e^{-i\lambda\xi}(1+\cos\tau)}{\frac{2\xi}{c} + 1 - \cos\tau} d\xi\, d\tau - A F_s \tag{3.2}$$

As was mentioned in section I-4, the mathematical procedure is simpler if the order of integration is reversed for the remaining integral of (3.2). Since the integral converges uniformly this reversal is permissible. With the help of algebraic manipulations this expression is put in the following form

$$\Gamma'' = -\frac{i\lambda A}{\pi}\int_0^{\infty}\int_0^{\pi} e^{-i\lambda\xi}\left[\frac{2\left(\frac{\xi}{c}+1\right)}{\frac{2\xi}{c} + 1 - \cos\tau} - 1\right] d\tau\, d\xi - A F_s$$

The second term in the brackets can be integrated at once, and the entire expression can be written as

$$\Gamma'' = -\frac{i\lambda A}{\pi}\int_0^{\infty} e^{-i\lambda\xi}\left[2\left(\frac{\xi}{c}+1\right)\int_0^{\pi}\frac{d\tau}{\frac{2\xi}{c}+1-\cos\tau} - \pi\right] d\xi - A F_s$$

The remaining integral in the brackets is an integral of the form

$$\int_0^{\pi}\frac{d\tau}{a - b\cos\tau} = \frac{-1}{\sqrt{a^2-b^2}} \arcsin\left[\frac{a\cos\tau - b}{a - b\cos\tau}\right]_0^{\pi}$$

$$= \frac{\pi}{\sqrt{a^2-b^2}}, \text{ for } a > b$$

where $a = \frac{2\xi}{c} + 1$ and $b = 1.$ Substituting this result, Γ'' becomes

$$\Gamma'' = -\frac{i\lambda A}{\pi}\int_0^{\infty} e^{-i\lambda\xi}\left[\frac{2\left(\frac{\xi}{c}+1\right)\pi}{\sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1}} - \pi\right] d\xi - A F_s$$

This result can now be reduced to the form

$$\Gamma'' = -i\lambda A\int_0^{\infty}\left[\sqrt{\frac{\xi + c}{\xi}} - 1\right] e^{-i\lambda\xi} d\xi - A F_s$$

As shown in appendix C,

$$\int_0^{\infty}\left[\sqrt{\frac{\xi+c}{\xi}} - 1\right] e^{-i\lambda\xi} d\xi = \frac{-1}{i\lambda}\left\{1 + \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda\xi}{2}}\, i H_0^{(2)}\left(\frac{\lambda c}{2}\right) + \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda\xi}{2}} H_1^{(2)}\left(\frac{\lambda c}{2}\right)\right\}$$

Here, $H_0^{(2)}(\frac{\lambda c}{2})$ is the so-called Hankel function of order zero, and likewise, $H_1^{(2)}(\frac{\lambda c}{2})$ is the Hankel function of order one, where $\frac{\lambda c}{2}$ is the variable of the function.

Since the above Hankel functions together with the coefficients occur and reoccur throughout this work, it is found convenient to symbolize these expressions. Thus Q_0 and Q_1 have been given the following meaning, i.e.,

$$Q_0 = \frac{\pi}{4} i \lambda c \, e^{\frac{i\lambda c}{2}} \, i \, H_0^{(2)}\left(\frac{\lambda c}{2}\right) \tag{3.3}$$

and

$$Q_1 = \frac{\pi}{4} i \lambda c \, e^{\frac{i\lambda c}{2}} \, H_1^{(2)}\left(\frac{\lambda c}{2}\right) \tag{3.4}$$

Using this notation, the expression for Γ'' can be put in its final form as

$$\Gamma'' = A(1 + Q_0 + Q_1) - A F_s \tag{3.5}$$

By examination of expression (3.1), it will be observed, that as the aspect ratio tends to infinity, the quantity F_s tends to zero. Expression (3.5) then takes the form

$$\Gamma'' = A(1 + Q_0 + Q_1)$$

since Q_0 and Q_1 are independent of aspect ratio. This form is the case of the two dimensional flutter theory where Q_0 and Q_1 are the basic functions in the flutter problem. These functions have been computed for several values of λc, and are given in Table 2.

TABLE 2

Real and imaginary values of Q_o and Q_1

$$Q_o = \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda c}{2}} i H_o^{(2)}\left(\tfrac{\lambda c}{2}\right) = -\frac{\pi \lambda c}{4}\left\{ J_o\left(\tfrac{\lambda c}{2}\right)\cos\tfrac{\lambda c}{2} + Y_o\left(\tfrac{\lambda c}{2}\right)\sin\tfrac{\lambda c}{2} + i\left[J_o\left(\tfrac{\lambda c}{2}\right)\sin\tfrac{\lambda c}{2} - Y_o\left(\tfrac{\lambda c}{2}\right)\cos\tfrac{\lambda c}{2}\right]\right\}$$

$$Q_1 = \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda c}{2}} H_1^{(2)}\left(\tfrac{\lambda c}{2}\right) = -\frac{\pi \lambda c}{4}\left\{ J_1\left(\tfrac{\lambda c}{2}\right)\sin\tfrac{\lambda c}{2} - Y_1\left(\tfrac{\lambda c}{2}\right)\cos\tfrac{\lambda c}{2} - i\left[J_1\left(\tfrac{\lambda c}{2}\right)\cos\tfrac{\lambda c}{2} + Y_1\left(\tfrac{\lambda c}{2}\right)\sin\tfrac{\lambda c}{2}\right]\right\}$$

λc	$\frac{\lambda c}{2}$	$Rl\ Q_o$	$Im\ Q_o$	$Rl\ Q_1$	$Im\ Q_1$
0.00	0.00	0.000000000	0.000000000	-1.00000000	0.0000000000
0.02	.01	-0.015234697	-0.047364299	-1.00021183	-0.0099239089
0.05	.025	-0.036874213	-0.096053281	-1.00104450	-0.024540339
0.10	.05	-0.070623145	-0.159183344	-1.00335559	-0.048244290
0.20	.10	-0.131844776	-0.25543628	-1.01028415	-0.093482976
0.30	.15	-0.18692035	-0.33107023	-1.01925685	-0.136223682
0.40	.20	-0.23734981	-0.39466031	-1.02960596	-0.176816578
0.50	.25	-0.28406176	-0.45009835	-1.04093752	-0.21552740
0.60	.30	-0.32769782	-0.49957271	-1.05299189	-0.25256733
0.70	.35	-0.38672994	-0.54446357	-1.06558684	-0.28810912
0.80	.40	-0.40752002	-0.58570740	-1.07858923	-0.32229711
0.90	.45	-0.44435397	-0.62397302	-1.09189888	-0.35525374
1.0	.50	-0.47946289	-0.65975666	-1.10543864	-0.38708417
2	1.0	-0.76608325	-0.93651853	-1.24466898	-0.65911803
3	1.5	-0.98417334	-1.13920159	-1.38003355	-0.87605414
4	2	-1.16525392	-1.30682253	-1.50756605	-1.05974251
5	2.5	-1.32278168	-1.45325970	-1.62733855	-1.22096359
6	3	-1.46381203	-1.58515410	-1.74015916	-1.36587543
7	3.5	-1.59252992	-1.70625636	-1.84689339	-1.49834119
8	4	-1.71163540	-1.81892409	-1.94831120	-1.62095104
9	4.5	-1.82298079	-1.92475378	-2.04506491	-1.73553076
10	5	-1.92789768	-2.02488716	-2.13769990	-1.84341878
200	100	-8.852143	-8.87501	-8.895604	-8.82914

II-4 Circulation due to Tip Trailing and Bound Vortices.

In section I-5, the downwash formula is given for the combined effects of the tip trailing and the bound vortices. It is given in that section as formula (5.5) and is

$$w''' = \frac{A}{2\pi c}\left[a_0 + a_1\left(\frac{x}{c}\right) + a_2\left(\frac{x}{c}\right)^2\right]$$

Transforming this to polar coordinates by means of formula (1.4) of section II-1, it becomes

$$w''' = \frac{A}{2\pi c}\left[a_0 + \frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{2}(a_1 + a_2)\cos\tau + \frac{a_2}{4}\cos^2\tau\right] \quad (4.1)$$

Substituting this for w in Munk's integral (1.2) of section II-1, it becomes

$$\Gamma''' = -\frac{A}{2\pi}\left\{\left(a_0 + \frac{a_1}{2} + \frac{a_2}{4}\right)\int_0^\pi d\tau + \left(a_0 - \frac{a_2}{4}\right)\int_0^\pi \cos\tau\, d\tau - \left(\frac{a_1}{2} + \frac{a_2}{4}\right)\int_0^\pi \cos^2\tau\, d\tau + \frac{a_2}{4}\int_0^\pi \cos^3\tau\, d\tau\right\}$$

where Γ''' designates the circulation about the wing due to the tip trailing and bound vortices alone. Performing the integration as indicated above, the result becomes

$$\Gamma''' = -\frac{A}{16}\left(8a_0 + 2a_1 + a_2\right) \quad (4.2)$$

Here again it is convenient to symbolize the above expression. For this purpose, the symbol F_c is introduced as follows:

$$F_c = \frac{1}{16}\left(8a_0 + 2a_1 + a_2\right) \quad (4.3)$$

The circulation Γ''' can now be written as

$$\Gamma''' = -A\, F_c \quad (4.4)$$

The quantity F_c is a function of the aspect ratio alone since $a_0, a_1,$ and a_2 depend only on aspect ratio. This is apparent if expressions (5.6), (5.7), and (5.8) of section I-5 are examined.

II-5 Circulation due to the Equivalent Wing Oscillation Downwash

The equivalent downwash due to the translatory and rotational oscillations of the wind is given by formula (6.6) of section I-6. This formula as there given is

$$w_0 = -\dot{h} - U\alpha - i\omega\alpha x$$

It is convenient to change the variable x, to τ. In changing this variable it must be remembered that the origin of x is taken at the mid-point of the chord, see figure 6.2 section I-6. In making this change formula (1.3) section II-1 must be used. From this it follows that

$$w_0 = -\dot{h} - U\alpha - \frac{i\omega c}{2}\alpha\cos\tau \tag{5.1}$$

Substituting this expression in Munk's integral, formula (1.2) section II-1, the circulation becomes

$$\Gamma^\circ = c\int_0^\pi\left(\dot{h} + U\alpha + \frac{i\omega c}{2}\alpha\cos\tau\right)(1+\cos\tau)\,d\tau$$

where Γ° designates the circulation due to w_0. On integration the above becomes

$$\Gamma^\circ = \pi c\left(\dot{h} + U\alpha + \frac{i\omega c}{4}\alpha\right) \tag{5.2}$$

II-6 Determination of the Total Circulation A

As was pointed out in section I-3, the downwash could not be computed explicitly since the circulation about the wing was unknown. For the purpose of computation the symbol A was introduced. This symbol is equal to the total circulation about the wing and it is now possible to write

$$A = \Gamma^{\circ} + \Gamma' + \Gamma'' + \Gamma''' \tag{6.1}$$

where the notation on the right side of this expression is explained in the preceding sections of this chapter. Applying expressions (2.7), (3.5), (4.3), and (5.1), of this chapter, the following expressions results for the total circulation

$$A = \Gamma^{\circ} - AF_T + A(1 + Q_0 + Q_1) - AF_S - AF_C$$

Solving this expression for A gives

$$A = \frac{\Gamma^{\circ}}{F_T + F_S + F_C - Q_0 - Q_1}$$

Here Γ° has been left in its symbolized form rather than introducing its equivalent as given by expression (5.1) of section II-5.

Another symbol will now be introduced in order to shorten the writting of the formula. Thus let

$$F = F_T + F_S + F_C \tag{6.2}$$

The expression for A can now be written as

$$A = \frac{\Gamma^{\circ}}{F - Q_0 - Q_1} \tag{6.3}$$

It is well to point out here, that as the aspect ratio tends to infinity, the quantity F tends to zero, and (6.3) reduces to the case of the two dimensional flutter problem. The truth of the above becomes evident if expressions (2.6), (3.1) and (4.3) of chapter II are examined together with expressions (5.6), (5.7), and (5.8), of chapter I.

Chapter III

LIFT AND MOMENT

III-1 Expressions for Lift and Moment in Terms of the Downwash

In order to obtain the forces acting on a wing, Euler's equations for fluid motion are used. Here however, only the x-component is used. This is given as equation (5.2) page 113, volume I of "Aerodynamic Theory", W. F. Durand editor-in-chief (see reference 1). Neglecting the body force, the equation can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1.1)$$

A further simplification can be made; it is called linearization. In chapter I, the velocity U of the undisturbed air stream was introduced. In the neighborhood of the wing, the air is disturbed and the velocity is no longer equal to U. The x-component of the air velocity can be written as $U+u$, where u takes on both positive and negative values. That part of the velocity designated by u is called the perturbation velocity, and is small compared to U. Since U is a constant, its derivative with respect to any variable is zero, then by replacing u by $U+u$ in equation (1.1), it becomes

$$\frac{\partial u}{\partial t} + (U+u)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Now since u, v, w, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$ are all small quantities, the products of any two of them will be neglected in the above equation. Since U is large however, the term $U\frac{\partial u}{\partial x}$ will be retained. Following this procedure the above expression becomes

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1.2}$$

This is the x-component of the linearized Euler equations for fluid motion.

To obtain the pressure on a wing, expression (1.2) must be integrated with respect to x. Consider the wing as a flat plate with the origin O at the midpoint of the chord EF as shown in figure 1.1. Let p_1 and p_2 designate the pressures acting on the upper and lower surfaces respectively at point x. Expression (1.2) is then integrated along the chord (the x-axis) for both the upper and lower surfaces.

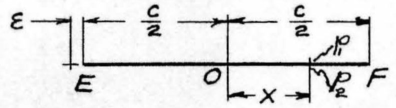

Fig. 1.1 - Flat Plate Wing

Since there exists a singularity at the leading edge (the point E) the integration of (1.2) is taken from $-\varepsilon-\frac{c}{2}$ to $\frac{c}{2}$; thus for the upper surface

$$\int_{-\varepsilon-\frac{c}{2}}^{x}\frac{\partial p_1}{\partial x}dx = -\rho\int_{-\varepsilon-\frac{c}{2}}^{x}\frac{\partial u_1}{\partial x}dx - \rho\frac{\partial}{\partial t}\int_{-\varepsilon-\frac{c}{2}}^{x}u_1(x)\,dx$$

or $$p_1(x)-p_1(-\varepsilon-\tfrac{c}{2}) = \rho U\left[u_1(x)-u_1(-\varepsilon-\tfrac{c}{2})\right]-\rho\frac{\partial}{\partial t}\int_{-\varepsilon-\frac{c}{2}}^{x}u_1(x)\,dx$$

and likewise for the lower surface, the result is

$$p_2(x)-p_2(-\varepsilon-\tfrac{c}{2}) = \rho U\left[u_2(x)-u_2(-\varepsilon-\tfrac{c}{2})\right]-\rho\frac{\partial}{\partial t}\int_{-\varepsilon-\frac{c}{2}}^{x}u_2(x)\,dx$$

Since the point $-\varepsilon-\frac{c}{2}$ is in front of the leading edge E it is evident that $p_1(-\varepsilon-\frac{c}{2})=p_2(-\varepsilon-\frac{c}{2})$ and $u_1(-\varepsilon-\frac{c}{2})=u_2(-\varepsilon-\frac{c}{2})$; hence the difference of the above two expressions become

$$p_2(x)-p_1(x) = -\rho U\left[u_2(x)-u_1(x)\right]-\rho\frac{\partial}{\partial t}\int_{-\varepsilon-\frac{c}{2}}^{x}\left[u_2(x)-u_1(x)\right]dx \tag{1.3}$$

where $p_2(x)-p_1(x)$ is the resultant pressure on the wing. Dropping the functional notation, multiplying (1.3) by dx and integrating from $-\frac{c}{2}$ to $\frac{c}{2}$ the lift per unit of span becomes

$$L = \int_{-\frac{c}{2}}^{\frac{c}{2}} (p_2 - p_1)\, dx$$

$$= -\rho U \int_{-\frac{c}{2}}^{\frac{c}{2}} (u_2 - u_1)\, dx - \rho \frac{\partial}{\partial t} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{\frac{c}{2}}^{x} (u_2 - u_1)\, dx \right] dx$$

In the above integrals it will be noticed that both integrands are the same, i.e., $u_2 - u_1$.

The quantity $u_2 - u_1$ or rather $(u_2 - u_1)dx$ can be shown to be the element of circulation as follows. In figure 1.2 the chord EF is shown and a small rectangle one side of which has the length dx. The other side is assumed to be negligibly small compared to dx so that it can be neglected. If positive circulation is that which augments lift, the circulation about the small rectangle shown in figure 1.2, must be taken in a counter-clockwise direction. This is based on the assumption that E is the leading edge, F is the trailing edge, the undisturbed airstream U is from left to right as shown on the figure, and that the lift is upward.

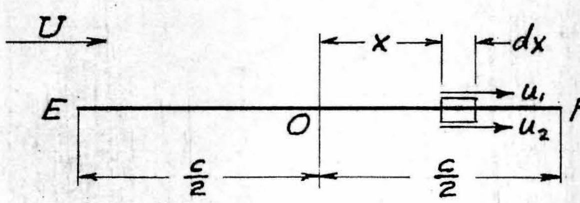

Fig. 1.2 - Element of Circulation

The circulation around this rectangular element is then equal to

$$u_1\, dx - u_2\, dx$$

If γ represents the strength of the ciruulation per unit of length, i.e., the vorticity, then

$$\gamma\, dx = u_1\, dx - u_2\, dx$$

or

$$-\gamma\, dx = (u_2 - u_1)\, dx$$

Substituting this into above expression for lift, it becomes

$$L = \rho U \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma \, dx + \rho \frac{\partial}{\partial t} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} \gamma \, dx \right] dx$$

The first integral is the total circulation about the wing, which is here designated by A and given by expression (6.3) section II-6. The second expression can be put in simpler form by an integration by parts. To begin with, let

$$F(x) = \int_{-\varepsilon-\frac{c}{2}}^{x} \gamma \, dx ,$$

then the second integral becomes

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} \gamma \, dx \right] dx = \int_{-\frac{c}{2}}^{\frac{c}{2}} F(x) \, dx$$

In the parts formula, i.e., $\int u \, dv = uv - \int v \, du$ let

$$u = F(x), \qquad dv = dx$$

then $\quad du = F'(x)\, dx = \gamma \, dx \qquad v = x$

Substituting the above, the integral becomes

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} \gamma \, dx \right] dx = x F(x) \Big]_{-\frac{c}{2}}^{\frac{c}{2}} - \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x \, dx$$

To evaluate the first term write

$$x F(x) = x \int_{-\varepsilon-\frac{c}{2}}^{x} \gamma \, dx$$

then when $x = \frac{c}{2}$

$$\frac{c}{2} F\left(\frac{c}{2}\right) = \underset{\varepsilon \to 0}{\text{LIMIT}} \frac{c}{2} \int_{-\varepsilon-\frac{c}{2}}^{\frac{c}{2}} \gamma \, dx$$

and when $x = -\frac{c}{2}$

$$-\frac{c}{2} F\left(-\frac{c}{2}\right) = -\underset{\varepsilon \to 0}{\text{LIMIT}} \frac{c}{2} \int_{-\varepsilon-\frac{c}{2}}^{-\frac{c}{2}} \gamma \, dx = 0$$

The first of the above integrals is the circulation about the wing, i.e., A; hence

$$\frac{c}{2}F\left(\frac{c}{2}\right) = \frac{c}{2}A \quad .$$

Substituting these results the equation for lift becomes

$$L = \rho U A + \rho\frac{\partial}{\partial t}\left[\frac{c}{2}A - \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x\, dx\right]$$

or

$$L = \rho U A + \frac{\rho c}{2}\frac{\partial A}{\partial t} - \rho\frac{\partial}{\partial t}\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x\, dx \tag{1.4}$$

Expression (1.4) involves a new integral, which for the purpose of abbreviation will be designated as B_2 ; thus

$$B_2 = \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x\, dx \tag{1.5}$$

The variable γ , which is the vorticity along the wing, can be expressed as a function of the downwash w . The expression is given by von Kármán and Burgers on page 46 of reference 2, and is rearranged in appendix D so that it appears as

$$\gamma = \frac{2}{\pi \sin\theta}\int_0^{\pi} \frac{w \sin^2\tau\, d\tau}{\cos\theta - \cos\tau} + \frac{2\Gamma}{\pi c \sin\theta} \tag{1.6}$$

Here $w = f(\tau)$ and the variable θ is related to the x given in the integral (1.5) by the expression

$$x = \frac{c}{2}\cos\theta$$

Substituting this value of x, and γ as given in expression (1.6), in integral (1.5) and integrating with respect to θ, it becomes

$$B_2 = \frac{c^2}{2}\int_0^{\pi} w \sin^2\tau\, d\tau \tag{1.7}$$

For the details of the above operations see appendix D.

Formula (1.4) can now be written completely in symbolized form by making use of (1.5), thus

$$L = \rho U A + \frac{\rho c}{2}\frac{\partial A}{\partial t} - \rho\frac{\partial B_2}{\partial t} \tag{1.8}$$

In the above expression occurs two differentiations with respect to time. The fact that A is a function of time is set forth in section I-3. Since the downwash w, depends on A, it is evident that B_2 is also a function of time. This property is reviewed in section III-3. In equation (1.8) A and B_2 are the instantaneous values, consequently the lift L is an instantaneous value.

The moment about the mid-point of the chord EF of figure 1.1 can be obtained, if equation (1.3) is multiplied by $x\,dx$ and integrated from the leading edge $(x=-\frac{c}{2})$ to the trailing edge $(x=\frac{c}{2})$. If figure 1.1 is examined it will be noticed that when x is positive, i.e., a point on line OF, and $p_2 - p_1$ is positive, the moment produced is a so-called diving moment. Since in aerodynamic theory a diving moment is usually taken negative, it will be advisable to introduce a minus sign along with $x\,dx$. The moment can now be written as

$$M = -\int_{-\frac{c}{2}}^{\frac{c}{2}} (p_2 - p_1)\, x\,dx$$

$$= \rho U \int_{\frac{c}{2}}^{\frac{c}{2}} (u_2 - u_1)\, x\,dx + \rho\frac{\partial}{\partial t}\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} (u_2 - u_1)\,dx\right] x\,dx$$

and since $-\gamma\,dx = (u_2 - u_1)\,dx$ or say $-\gamma = u_2 - u_1$, the above can be written as

$$M = -\rho U \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x\,dx - \rho\frac{\partial}{\partial t}\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} \gamma\,dx\right] x\,dx$$

of by means of expression (1.5)

$$M = -\rho U B_2 - \rho\frac{\partial}{\partial t}\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} \gamma\,dx\right] x\,dx$$

As in the lift equation, the above expression can be reduced by placing

$$F(x) = \int_{-\varepsilon-\frac{c}{2}}^{x} \gamma \, dx$$

and integrating by parts using the formula $\int u\,dv = uv - \int v\,du$. For this purpose let

$$u = F(x), \qquad dv = x\,dx,$$

then

$$du = F'(x)\,dx = \gamma\,dx, \qquad v = \frac{x^2}{2}.$$

The integral in the expression for the moment can now be written as

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{x} \gamma\,dx \right] x\,dx = \int_{-\frac{c}{2}}^{\frac{c}{2}} F(x)\,x\,dx$$

$$= \frac{x^2}{2} F(x) \Big]_{-\frac{c}{2}}^{\frac{c}{2}} - \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma \frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{c^2}{4} F\left(\frac{c}{2}\right) - \frac{c^2}{4} F\left(-\frac{c}{2}\right) \right] - \frac{1}{2}\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2\,dx$$

As was shown in this section preceding expression (1.4)

$$F\left(\frac{c}{2}\right) = \underset{\varepsilon \to 0}{\text{LIMIT}} \int_{-\varepsilon-\frac{c}{2}}^{\frac{c}{2}} \gamma\,dx = A$$

and

$$F\left(-\frac{c}{2}\right) = \underset{\varepsilon \to 0}{\text{LIMIT}} \int_{-\varepsilon-\frac{c}{2}}^{-\frac{c}{2}} \gamma\,dx = 0$$

substituting these values the integral becomes

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\int_{-\varepsilon-\frac{c}{2}}^{\frac{c}{2}} \gamma\,dx \right] x\,dx = \frac{c^2}{8} A - \frac{1}{2} \int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2\,dx$$

The expression for the moment becomes on substitution of the above values the following;

$$M = -\rho U B_2 - \rho \frac{\partial}{\partial t}\left[\frac{c^2}{8}A - \frac{1}{2}\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx\right]$$

$$= -\rho U B_2 - \rho \frac{c^2}{8}\frac{\partial A}{\partial t} - \frac{1}{2}\rho \frac{\partial}{\partial t}\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx \qquad (1.9)$$

Attention is called to the fact that the above expression is for the instantaneous value of the moment, or in other words, the value at some particular instant of time.

Expression (1.9) also involves a new integral, which is

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx \qquad (1.10)$$

Using expression (1.6) and the relation $x = \frac{c}{2}\cos\theta$, integral (1.10) can be expressed in terms of the downwash; thus

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx = \frac{c^3}{4}\int_0^{\pi} w \sin^2\tau \cos\tau \, d\tau + \frac{c^2\Gamma}{8} \qquad (1.11)$$

The details of this transformation are given in appendix D. For the purpose of abreviation the symbol B_3 is introduced, and is defined as follows; thus let

$$B_3 = \frac{c^3}{4}\int_0^{\pi} w \sin^2\tau \cos\tau \, d\tau \qquad (1.12)$$

Since Γ represents the total circulation about the wing, which is herein designated as A, integral (1.11) can now be written in the form as given below, i.e.,

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx = B_3 + \frac{c^2 A}{8}$$

Substituting this in expression (1.9) it becomes

$$M = -\rho U B_2 - \frac{\rho c^2}{8}\frac{\partial A}{\partial t} + \frac{\rho}{2}\frac{\partial}{\partial t}\left[B_3 + \frac{c^2 A}{8}\right]$$

which reduces to

$$M = -\rho U B_2 - \frac{\rho c^2}{16}\frac{\partial A}{\partial t} + \frac{\rho}{2}\frac{\partial B_3}{\partial t} \tag{1.13}$$

Mention has been made of the fact that A and B_2 are functions of time. This is also true of B_3 and is set forth in section III-3. The moment given by expression (1.13) is the instantaneous value.

III-2 Evaluation of the expressions for Lift and Moment

In the integrals (1.7) and (1.12), of section III-1 it is to be remembered that

$$w = w' + w'' + w''' + w_0$$

With this expression integral (1.7) can be written as

$$B_2 = \frac{C^2}{2}\int_0^{\pi}(w' + w'' + w''' + w_0)\sin^2\tau \, d\tau$$

It is convenient to separate the above into four integrals. In order to designate these four the following notation is used;

$$B_2' = \frac{C^2}{2}\int_0^{\pi} w' \sin^2\tau \, d\tau \qquad (2.1)$$

$$B_2'' = \frac{C^2}{2}\int_0^{\pi} w'' \sin^2\tau \, d\tau \qquad (2.2)$$

$$B_2''' = \frac{C^2}{2}\int_0^{\pi} w''' \sin^2\tau \, d\tau \qquad (2.3)$$

$$B_2^{\circ} = \frac{C^2}{2}\int_0^{\pi} w_0 \sin^2\tau \, d\tau \qquad (2.4)$$

From this it follows that

$$B_2 = B_2' + B_2'' + B_2''' + B_2^{\circ} \qquad (2.5)$$

Integral (1.12) is handled in like manner, thus it also can be written as

$$B_3 = \frac{C^3}{4}\int_0^{\pi}(w' + w'' + w''' + w_0)\sin^2\tau \cos\tau \, d\tau$$

and also a like notation can be defined as

$$B_3' = \frac{C^3}{4}\int_0^{\pi} w' \sin^2\tau \cos\tau \, d\tau \qquad (2.6)$$

$$B_3'' = \frac{C^3}{4}\int_0^{\pi} w'' \sin^2\tau \cos\tau \, d\tau \qquad (2.7)$$

$$B_3''' = \frac{C^3}{4}\int_0^{\pi} w''' \sin^2\tau \cos\tau \, d\tau \qquad (2.8)$$

$$B_3^{\circ} = \frac{C^3}{4}\int_0^{\pi} w_0 \sin^2\tau \cos\tau \, d\tau \qquad (2.9)$$

where

$$B_3 = B_3' + B_3'' + B_3''' + B_3^\circ \tag{2.10}$$

The above integrals will be evaluated in this section in the same order as they are numbered. To begin consider integral (2.1). The downwash w' is given in section I-3 as expression (3.4) but since it is desirable to have w' expressed as a function of τ, the form as given in section II-2 will be used. On substitution of this form, integral (2.1) becomes

$$B_2' = \frac{C^2}{2}\int_0^\pi \frac{A e^{-\frac{c}{r_1}}}{2\pi r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1} e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left[e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right]\right\}\sin^2\tau\, d\tau$$

To bring this into form substitute for $\sin^2\tau$ the trigonometric identity

$$\sin^2\tau = \tfrac{1}{2}(1-\cos 2\tau)$$

and for $\cos\tau\cos 2\tau$ use the identity

$$\cos\tau\cos 2\tau = \tfrac{1}{2}(\cos\tau + \cos 3\tau)$$

The integral for B_2 can now be written as

$$B_2' = \frac{Ac^2 e^{-\frac{c}{r_1}}}{8\pi r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1}\int_0^\pi (1-\cos 2\tau)\, e^{\frac{c}{r_1}\cos\tau} d\tau + \frac{c}{r_1}\int_0^\pi\left(1-\tfrac{1}{2}\cos\tau - \cos 2\tau + \tfrac{1}{2}\cos 3\tau\right)e^{\frac{c}{r_1}\cos\tau} d\tau\right\}$$

In this expression it will be noticed, that there are four distinct types of integrals, the values of which can be given at once by means of integral (B1) of appendix B. These four types with their values are

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\,d\tau = \pi I_0\left(\tfrac{c}{r_1}\right)$$

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos\tau\,d\tau = \pi I_1\left(\tfrac{c}{r_1}\right)$$

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos 2\tau\,d\tau = \pi I_2\left(\tfrac{c}{r_1}\right)$$

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau}\cos 3\tau\,d\tau = \pi I_3\left(\tfrac{c}{r_1}\right)$$

Substituting these values in the expression for B_2' and canceling the π, it becomes

$$B_2' = \frac{Ac^2e^{-\frac{c}{r_1}}}{8r_1(2+i\lambda r_1)}\left\{\frac{4+i\lambda r_1}{2+i\lambda r_1}\left[I_0\left(\tfrac{c}{r_1}\right) - I_2\left(\tfrac{c}{r_1}\right)\right]\right.$$
$$\left. + \frac{c}{r_1}\left[I_0\left(\tfrac{c}{r_1}\right) - \tfrac{1}{2}I_1\left(\tfrac{c}{r_1}\right) - I_2\left(\tfrac{c}{r_1}\right) + \tfrac{1}{2}I_3\left(\tfrac{c}{r_1}\right)\right]\right\}$$

The modified Bessel functions in this expression of order greater than one can be reduced to orders zero and one by means of identities (B3) and (B4) given in appendix B. The fraction preceding the first brackets in the expression for B_2' can be separated into two terms, one being unity and the other fraction containing $i\lambda r_1$ in the denominator. Performing these operations the expression can be written as

$$B_2' = \frac{Ac^2e^{-\frac{c}{r_1}}}{8r_1(2+i\lambda r_1)}\left\{\left[\frac{2}{2+i\lambda r_1} + 1\right]\left[I_0\left(\tfrac{c}{r_1}\right) - I_0\left(\tfrac{c}{r_1}\right) + \tfrac{2r_1}{c}I_1\left(\tfrac{c}{r_1}\right)\right]\right.$$
$$+ \frac{c}{r_1}\left[I_0\left(\tfrac{c}{r_1}\right) - \tfrac{1}{2}I_1\left(\tfrac{c}{r_1}\right) - I_0\left(\tfrac{c}{r_1}\right) + \tfrac{2r_1}{c}I_1\left(\tfrac{c}{r_1}\right) - \tfrac{2r_1}{c}I_0\left(\tfrac{c}{r_1}\right)\right.$$
$$\left.\left. + \left(\tfrac{1}{2} + \tfrac{4r_1^2}{c^2}\right)I_1\left(\tfrac{c}{r_1}\right)\right]\right\}$$

which reduces to

$$B_2' = \frac{A c^2 e^{-\frac{c}{r_1}}}{4 r_1 (2 + i\lambda r_1)} \left\{ \frac{1}{2 + i\lambda r_1} \frac{2 r_1}{c} I_1\left(\frac{c}{r_1}\right) - I_0\left(\frac{c}{r_1}\right) + \left(1 + \frac{3 r_1}{c}\right) I_1\left(\frac{c}{r_1}\right) \right\}$$

Using expression (2.2) of section I-2, i.e., $r_1 = \dfrac{c\,AR}{4}$, the above expression can be put in its final form:

$$B_2' = \frac{A c\, e^{-\frac{4}{AR}}}{1 + \frac{i\lambda c AR}{8}} \left\{ \frac{\frac{1}{8} I_1\left(\frac{4}{AR}\right)}{1 + \frac{i\lambda c AR}{8}} - \frac{1}{2AR} I_0\left(\frac{4}{AR}\right) + \left(\frac{1}{2AR} + \frac{3}{8}\right) I_1\left(\frac{4}{AR}\right) \right\} \tag{2.11}$$

The symbol F_{1T} is now introduced and is defined as follows:

$$F_{1T} = \frac{e^{-\frac{4}{AR}}}{1 + \frac{i\lambda c AR}{8}} \left\{ \frac{\frac{1}{8} I_1\left(\frac{4}{AR}\right)}{1 + \frac{i\lambda c AR}{8}} - \frac{1}{2AR} I_0\left(\frac{4}{AR}\right) + \left(\frac{1}{2AR} + \frac{3}{8}\right) I_1\left(\frac{4}{AR}\right) \right\} \tag{2.12}$$

This symbol, as well as the F's introduced in chapter II, is without dimension. Using this symbol B_2' can be expressed as

$$B_2' = A c F_{1T} \tag{2.13}$$

Next in order is integral (2.2). The expression for the downwash w'' is taken from the first paragraph of section II-3. Substituting this value in expression (2.2) it becomes:

$$B_2'' = \frac{c^2}{2} \int_0^{\pi} \left\{ \frac{i\lambda A}{\pi c} \int_0^{\infty} \frac{e^{-i\lambda\xi}\, d\xi}{\frac{2\xi}{c} + 1 - \cos\tau} - \frac{1.1358\, i\lambda A}{\pi(1 + 6 i\lambda r_1)} e^{-\frac{c}{12 r_1}(1-\cos\tau)} \right\} \sin^2\tau\, d\tau$$

As is to be seen, the first integral of the right member of the above expression is a double integral, while the second term involves only an integration with respect to τ . As was done in the integration for Γ'' given section II-3, the order of integration will be reversed for the first term.

Since the entire right member of the above expression is multiplied by $\sin^2\tau$, the first term of this expression will be multiplied by $1-\cos^2\tau$ and the second by $\frac{1}{2}(1-\cos 2\tau)$. The expression for B_2 can now be arranged as shown below:

$$B_2'' = \frac{i\lambda c}{2\pi} A \int_0^\infty \int_0^\pi \frac{(\cos^2\tau - 1)\, e^{-i\lambda\xi}}{\cos\tau - \left(\frac{2\xi}{c}+1\right)}\, d\tau\, d\xi$$

$$- \frac{1.1358\, i\lambda c^2}{4\pi(1+6i\lambda r_1)} A\, e^{-\frac{c}{12 r_1}} \int_0^\pi e^{\frac{c}{12 r_1}\cos\tau} (1-\cos 2\tau)\, d\tau$$

Since the second integral in this expression can be evaluated by means of integral (B1) of appendix B, it will be treated first. If $\frac{c}{r_1}$ is replaced by $\frac{c}{12 r_1}$ in integral (B1), the second integral occurring in B_2'' can be written as

$$\int_0^\pi e^{\frac{c}{12 r_1}\cos\tau}(1-\cos 2\tau)\, d\tau = \pi I_0\left(\tfrac{c}{12 r_1}\right) - \pi I_2\left(\tfrac{c}{12 r_1}\right)$$

and making use of identity (B6), given in appendix B, it becomes

$$\int_0^\pi e^{\frac{c}{12 r_1}\cos\tau}(1-\cos 2\tau)\, d\tau = \frac{24\pi r_1}{c} I_1\left(\tfrac{c}{12 r_1}\right)$$

Substituting this in the expression for B_2'', gives

$$B_2'' = \frac{i\lambda c}{2\pi} A \int_0^\infty \int_0^\pi \frac{(\cos^2\tau - 1)\, e^{-i\lambda\xi}}{\cos\tau - \left(\frac{2\xi}{c}+1\right)}\, d\tau\, d\xi$$

$$- \frac{6.8148\, i\lambda c r_1}{1+6i\lambda r_1} A\, e^{-\frac{c}{12 r_1}} I_1\left(\tfrac{c}{12 r_1}\right)$$

A symbol for the second term will now be defined as follows. Thus let

$$F_{1s} = -\frac{6.8148\, i\lambda r_1}{1+6i\lambda r_1} e^{-\frac{c}{12 r_1}} I_1\left(\tfrac{c}{12 r_1}\right)$$

This can be reduced by means of expression (2.2) section I-2 to appear as shown below:

$$F_{1s} = -\frac{1.7037\, i\lambda c\, AR}{1+\frac{3}{2} i\lambda c\, AR}\, e^{-\frac{1}{3AR}}\, I_1\left(\frac{1}{3AR}\right) \qquad (2.14)$$

Before substituting F_{1s} in the expression for B_2'', attention is called to the fact that the integrand of the first integral is an improper fraction. In view of this fact the fraction can be put in proper form by long division and B_2'' can be written as

$$B_2'' = \frac{i\lambda c}{2\pi} A \int_0^\infty \int_0^\pi \left[\cos\tau + \left(\frac{2\xi}{c}+1\right) - \frac{\left(\frac{2\xi}{c}+1\right)^2 - 1}{\left(\frac{2\xi}{c}+1\right) - \cos\tau}\right] e^{-i\lambda\xi}\, d\tau\, d\xi + AcF_{1s}$$

In this expression the integral of the first term is zero, and integral of the second is $\left(\frac{2\xi}{c}+1\right)\pi$. The integral of the third term however, is of the same type as was encountered in section II-3, i.e.,

$$\int_0^\pi \frac{d\tau}{a - b\cos\tau} = \frac{\pi}{\sqrt{a^2+b^2}}$$

where $a > b$. Here $a = \frac{2\xi}{c} + 1$ and $b = 1$, hence canceling the π resulting from the integration, the expression for B_2'' becomes

$$B_2'' = \frac{i\lambda c}{2} A \int_0^\infty \left[\frac{2\xi}{c} + 1 - \frac{\left(\frac{2\xi}{c}+1\right)^2 - 1}{\sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1}}\right] e^{-i\lambda\xi}\, d\xi + AcF_{1s}$$

or

$$B_2'' = \frac{i\lambda c}{2} A \int_0^\infty \left[\frac{2\xi}{c} + 1 - \sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1}\right] e^{-i\lambda\xi}\, d\xi + AcF_{1s} \qquad (2.15)$$

Using integral $(E3)$ of appendix E, the above can be put in the following form:

$$B_2'' = \frac{i\lambda c}{2} A \left[\frac{1}{i\lambda} - \frac{2}{\lambda^2 c} + \frac{\pi}{2}\frac{e^{\frac{i\lambda c}{2}}}{i\lambda} H_1^{(2)}\left(\frac{\lambda c}{2}\right)\right] + AcF_{1s}$$

$$= Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{\pi}{4} e^{\frac{i\lambda c}{2}} H_1^{(2)}\left(\frac{\lambda c}{2}\right)\right] + AcF_{1s}$$

This equation can be further reduced by making use of the symbol Q_1 as defined by expression (3.4) of section II-3; thus

$$B_2'' = Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_{1s}\right] \tag{2.16}$$

Next in order is integral (2.3). For w''', the form given as expression (4.1) in section II-4 will be used. Using this expression integral (2.3) becomes

$$B_2''' = \frac{c^2}{2}\int_0^{\pi} \frac{A}{2\pi c}\left[a_0 + \frac{a_1}{2} + \frac{a_2}{4} - \frac{1}{2}(a_1 + a_2)\cos\tau + \frac{a_2}{4}\cos^2\tau\right]\sin^2\tau\, d\tau$$

To integrate the above, replace $\sin^2\tau$ by $1-\cos^2\tau$, and multiply each term of the expression by this factor. Since the technique of this integration is quite simple, it will be omitted, and the final result will be written without comment; thus

$$B_2''' = \frac{Ac}{128}\left[16a_0 + 8a_1 + 5a_2\right] \tag{2.17}$$

This will be symbolized by the letter F_{1c} ; thus

$$F_{1c} = \frac{1}{128}\left[16a_0 + 8a_1 + 5a_2\right] \tag{2.18}$$

so that

$$B_2''' = Ac F_{1c} \tag{2.19}$$

The last of the B_2's is the integral (2.4). Expression (5.1) section II-5 is used for w_0 in this calculation. With this integral (2.4) becomes

$$B_2^{\circ} = \frac{c^2}{2}\int_0^{\pi}\left[-\dot{h} - U\alpha - \frac{i\omega c}{2}\alpha\cos\tau\right]\sin^2\tau\, d\tau$$

which integrates at once as

$$B_2^{\circ} = -\frac{\pi c^2}{4}\left[\dot{h} + U\alpha\right] \tag{2.20}$$

No symbolization other than that given above will be introduced for expression (2.20). For abbreviation, the symbol B_2° can be used.

Expression (2.5) can now be written by assembling expressions (2.13), (2.16), (2.19), and (2.20); thus

$$B_2 = AcF_{1T} + Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_{1S}\right] + AcF_{1C} - \frac{\pi c^2}{4}\left[\dot{h} + U\alpha\right]$$

The symbol F_1 will be introduced such that

$$F_1 = F_{1T} + F_{1S} + F_{1C} \tag{2.21}$$

and with it B_2 becomes,

$$B_2 = -\frac{\pi c^2}{4}\left[\dot{h} + U\alpha\right] + Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_1\right] \tag{2.22}$$

or simply

$$B_2 = B_2^\circ + Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_1\right] \tag{2.23}$$

Attention is called to the fact, that all the above quantities are known for any particular wing, when the air speed and the frequency of the wing's oscillations are known. Thus A, is given by expression (6.3) of section II-6, and the relation between ω and λ is given by expression (6.3) of section I-6.

The next integral is given by expression (2.6). Here again the same downwash formula for w' will be used, i.e. the form given in the first paragraph of section II-3, which is

$$w' = \frac{A e^{-\frac{c}{r_1}}}{2\pi r_1(2 + i\lambda r_1)}\left\{\frac{4 + i\lambda r_1}{2 + i\lambda r_1} e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left[e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau\right]\right\}$$

Substituting in expression (2.6) gives

$$B_3' = \frac{A c^3 e^{-\frac{c}{r_1}}}{8\pi r_1(2+i\lambda r_1)} \int_0^\pi \left\{ \frac{4+i\lambda r_1}{2+i\lambda r_1} e^{\frac{c}{r_1}\cos\tau} + \frac{c}{r_1}\left[e^{\frac{c}{r_1}\cos\tau} - e^{\frac{c}{r_1}\cos\tau}\cos\tau \right] \right\} \sin^2\tau\cos\tau\, d\tau$$

By means of the trigonometric identities, the above integral can be rearranged to appear as shown below:

$$B_3' = \frac{A c^3 e^{-\frac{c}{r_1}}}{32\pi r_1(2+i\lambda r_1)} \left\{ \frac{4+i\lambda r_1}{2+i\lambda r_1} \int_0^\pi e^{\frac{c}{r_1}\cos\tau}(\cos\tau - \cos 3\tau)\, d\tau + \frac{c}{r_1}\int_0^\pi e^{\frac{c}{r_1}\cos\tau}\left(-\frac{1}{2} + \cos\tau - \cos 3\tau + \frac{1}{2}\cos 4\tau\right) d\tau \right\}$$

The integration involved in this expression is again an application of integral $(B1)$, of appendix B. Applying this integral, and canceling the π the expression for B_3' becomes

$$B_3' = \frac{A c^3 e^{-\frac{c}{r_1}}}{32 r_1(2+i\lambda r_1)} \left\{ \frac{4+i\lambda r_1}{2+i\lambda r_1}\left[I_1\left(\frac{c}{r_1}\right) - I_3\left(\frac{c}{r_1}\right)\right] + \frac{c}{r_1}\left[-\frac{1}{2} I_0\left(\frac{c}{r_1}\right) + I_1\left(\frac{c}{r_1}\right) - I_3\left(\frac{c}{r_1}\right) + \frac{1}{2} I_4\left(\frac{c}{r_1}\right)\right]\right\}$$

If r_1 is eliminated by means of expression (2.2) section I-2, and the modified Bessel functions are reduced by means of identities $(B3)$, $(B4)$, and $(B5)$ of appendix B, the above expression can be rearranged algebraically so that

$$B_3' = \frac{A c^2 e^{-\frac{4}{R}}}{16\left(1+\frac{i\lambda c R}{8}\right)} \left\{ \frac{1}{1+\frac{i\lambda c R}{8}}\left[I_0\left(\frac{4}{R}\right) - \frac{R}{2} I_1\left(\frac{4}{R}\right)\right] + \frac{4}{R}\left[(1+R) I_0\left(\frac{4}{R}\right) - \left(1+\frac{R}{2}+\frac{R^2}{2}\right) I_1\left(\frac{4}{R}\right)\right]\right\} \quad (2.24)$$

For this expression a symbol F_{0T} will be established; the definition of which is given below:

$$F_{0T} = \frac{e^{-\frac{4}{AR}}}{16\left(1+\frac{i\lambda c AR}{8}\right)}\left\{\frac{1}{1+\frac{i\lambda c AR}{8}}\left[I_0\left(\tfrac{4}{AR}\right) - \tfrac{AR}{2} I_1\left(\tfrac{4}{AR}\right)\right]\right.$$

$$\left. + \frac{4}{AR}\left[(1+AR)I_0\left(\tfrac{4}{AR}\right) - \left(1+\tfrac{AR}{2}+\tfrac{AR^2}{2}\right) I_1\left(\tfrac{4}{AR}\right)\right]\right\} \qquad (2.25)$$

From this, B_3' can be written as

$$B_3' = A c^2 F_{0T} \qquad (2.26)$$

To evaluate integral (2.7), the downwash w'' is taken from the first paragraph of II-3. Substituting this value of w'' in integral (2.7), the expression for B_3'' becomes

$$B_3'' = \frac{c^3}{4}\int_0^{\pi}\left\{\frac{i\lambda A}{\pi c}\int_0^{\infty}\frac{e^{-i\lambda\xi}\,d\xi}{\frac{2\xi}{c}+1-\cos\tau}\right.$$

$$\left. - \frac{1.1358\, i\lambda A}{\pi(1+6i\lambda \mathcal{R}_1)}\, e^{-\frac{c}{12\mathcal{R}_1}(1-\cos\tau)}\right\}\sin^2\tau\cos\tau\, d\tau$$

For the first term of the right member $\sin^2\tau\cos\tau$ will be replaced by its equivalent $\cos\tau - \cos^3\tau$, and in the second term by $\frac{1}{4}(\cos\tau - \cos 3\tau)$. If the order of integration of the first term is reversed, B_3'' can be written as

$$B_3'' = \frac{i\lambda c^2 A}{4\pi}\int_0^{\infty}\int_0^{\pi}\frac{\cos\tau - \cos^3\tau}{\frac{2\xi}{c}+1-\cos\tau}\, e^{-i\lambda\xi}\, d\tau\, d\xi$$

$$- \frac{1.1358\, i\lambda c^3 A}{16\pi(1+6i\lambda\mathcal{R}_1)}\, e^{-\frac{c}{12\mathcal{R}_1}}\int_0^{\pi} e^{\frac{c}{12\mathcal{R}_1}\cos\tau}(\cos\tau - \cos 3\tau)\, d\tau$$

The integral of the second term, being the simpler, will be evaluated first, This can be done by means of formula (B1) of appendix B. By means of identity (B7), the result can be given in terms of the modified

Bessel functions of orders zero and one. The numerator and denominator of the first integrand will be multiplied by minus one. Performing these operations, B_3'' takes the form

$$B_3'' = \frac{i\lambda c^2 A}{4\pi}\int_0^\infty\int_0^\pi \frac{\cos^3\tau - \cos\tau}{\cos\tau - \left(\frac{2\xi}{c}+1\right)} e^{-i\lambda\xi}\, d\tau\, d\xi$$

$$- \frac{1.1358\, i\lambda c^3 A}{16(1+6i\lambda r_1)} e^{-\frac{c}{12r_1}}\left[\frac{48 r_1}{c} I_0\left(\frac{c}{12r_1}\right) - 1152\frac{r_1^2}{c^2} I_1\left(\frac{c}{12 r_1}\right)\right]$$

where the π has been canceled in the second term.

At this point it is convenient to define a symbol F_{os} such that

$$F_{os} = -\frac{1.1358\, i\lambda c}{16(1+6i\lambda r_1)} e^{-\frac{c}{12r_1}}\left[\frac{48 r_1}{c} I_0\left(\frac{c}{12r_1}\right) - 1152\frac{r_1^2}{c^2} I_1\left(\frac{c}{12 r_1}\right)\right]$$

Applying expression (2.2) of section I-2, i.e., $r_1 = \dfrac{cR}{4}$, the above formula can be reduced to

$$F_{os} = -\frac{0.85185\, i\lambda c R}{1+\frac{3}{2} i\lambda c R} e^{-\frac{1}{3R}}\left[I_0\left(\frac{1}{3R}\right) - 6R\, I_1\left(\frac{1}{3R}\right)\right] \tag{2.27}$$

The integrand of the first integral of B_3'' is an improper fraction. It will now be made a proper fraction by algebraic long division. Performing this operation and substituting F_{os}, the expression for B_3'' can be made to appear as shown below:

$$B_3'' = \frac{i\lambda c^2 A}{4\pi}\int_0^\infty\int_0^\pi \Bigg\{\cos^2\tau + \left(\frac{2\xi}{c}+1\right)\cos\tau + \left[\left(\frac{2\xi}{c}+1\right)^2 - 1\right]$$

$$- \frac{\left(\frac{2\xi}{c}+1\right)\left[\left(\frac{2\xi}{c}+1\right)^2 - 1\right]}{\left(\frac{2\xi}{c}+1\right) - \cos\tau}\Bigg\} e^{-i\lambda\xi}\, d\tau\, d\xi + A c^2 F_{os}$$

Carrying out the integration with respect to τ, the first term of the above integral, i.e., $\cos^2\tau$ evaluates as $\frac{\pi}{2}$, the second term as zero, and the third as the bracketed factor times π. The last term is an integral of the form

$$\int_0^{\pi} \frac{d\tau}{a - b\cos\tau} = \frac{\pi}{\sqrt{a^2 - b^2}}$$

which was first introduced in section II-3. As in that section, $a = \frac{2\xi}{c} + 1$ and $b = 1$. Substituting the above integrations, and canceling π with the one in the prefactor, B_3'' becomes

$$B_3'' = \frac{i\lambda c^2 A}{4} \int_0^{\infty} \left\{ \frac{1}{2} + \left[\left(\frac{2\xi}{c} + 1 \right)^2 - 1 \right] - \left(\frac{2\xi}{c} + 1 \right) \sqrt{\left(\frac{2\xi}{c} + 1 \right)^2 - 1} \right\} e^{-i\lambda\xi} d\xi + A c^2 F_{os} \qquad (2.28)$$

The details of the remaining integration are given in appendix E. Using the result there given, i.e., expression (E7), the above formula becomes

$$B_3'' = Ac^2 \left\{ \frac{1}{8} + \frac{1}{i\lambda c} - \frac{2}{\lambda^2 c^2} + \frac{\pi}{8} e^{\frac{i\lambda c}{2}} i H_0^{(2)}\left(\frac{\lambda c}{2}\right) + \frac{\pi}{8} \frac{e^{\frac{i\lambda c}{2}}}{i\lambda c} H_1^{(2)}\left(\frac{\lambda c}{2}\right) \right\} + Ac^2 F_{os} \qquad (2.29)$$

Making use of the notation defined by expressions (3.3) and (3.4) of section II-3, B_3'' can be put in its final form as shown below:

$$B_3'' = Ac^2 \left\{ \frac{1}{8} + \frac{1}{i\lambda c} - \frac{2}{\lambda^2 c^2} + \frac{Q_0}{2i\lambda c} - \frac{2Q_1}{\lambda^2 c^2} \right\} + Ac^2 F_{os} \qquad (2.30)$$

To integrate expression (2.8), substitute for w'' formula (4.1) of section (II-4). Thus

$$B_3''' = \frac{c^3}{4}\int_0^\pi \frac{A}{2\pi c}\Big[\left(a_0 + \frac{a_1}{2} + \frac{a_2}{4}\right) - \frac{1}{2}(a_1 + a_2)\cos\tau + \frac{a_2}{4}\cos^2\tau\Big]\sin^2\tau\cos\tau\, d\tau$$

The integration of the above expression is carried out by first replacing $\sin^2\tau$ by $1-\cos^2\tau$, and then multiplying each term in the bracketed expression by the resulting factor. On collecting terms, the integration involves all positive integral powers of $\cos\tau$ as high as the fifth. Since the integrations are elementary, the details will not be carried out here. The final result of the above expression is

$$B_3''' = -\frac{Ac^2}{128}(a_1 + a_2) \tag{2.31}$$

and the symbol F_{0c} is adopted to mean

$$F_{0c} = -\frac{1}{128}(a_1 + a_2) \tag{2.32}$$

Using this symbol B_3''' can be written as

$$B_3''' = Ac^2 F_{0c} \tag{2.33}$$

The last integral to evaluate of this sequence is B_3^0. It is given by expression (2.9). For w_0 use equation (5.1) section II-5, and on substitution integral (2.9) becomes

$$B_3^0 = \frac{c^3}{4}\int_0^\pi \left(-\dot{h} - U\alpha - \frac{i\omega c}{2}\alpha\cos\tau\right)\sin^2\tau\cos\tau\, d\tau$$

As a result of the integration the first two terms in the parenthesis vanish, and the last term gives the result that

$$B_3''' = -\frac{\pi}{64} i\omega c^4 \alpha \tag{2.34}$$

No symbolization will be introduced for this expression. Attention is again called to relation (6.3) given in section I-6, i.e., $\omega = \lambda U$

Expression (2.10) can now be assembled. Using expressions (2.26), (2.30), (2.33) and (2.34), B_3 can be written as

$$B_3 = Ac^2 F_{oT} + Ac^2\left\{\frac{1}{8} + \frac{1}{i\lambda c} - \frac{2}{\lambda^2 c^2} + \frac{Q_0}{2i\lambda c} - \frac{2Q_1}{\lambda^2 c^2}\right\} + Ac^2 F_{os} + Ac^2 F_{oc} + \frac{\pi}{64} i\omega c^4 \alpha$$

As in the case of B_2, a symbol will be introduced such that

$$F_o = F_{oT} + F_{os} + F_{oc} \qquad (2.35)$$

With this symbol, and replacing $\frac{\pi}{64} i\omega c^4 \alpha$ by its symbol B_3^o, the expression for B_3 can be written as

$$B_3 = B_3^o + Ac^2\left\{\frac{1}{8} + \frac{1}{i\lambda c} - \frac{2}{\lambda^2 c^2} + \frac{Q_0}{2i\lambda c} - \frac{2Q_1}{\lambda^2 c^2} + F_o\right\} \qquad (2.36)$$

III-3 Final Form of Lift and Moment Formulae

In section I-3 the instantaneous nature of the circulation A, was pointed out. Likewise the quantities B_2 and B_3 defined by expressions (1.7) and (1.12) section III-1 respectively, are the instantaneous values. The three above mentioned quantities can be written as

$$A = \bar{A} e^{i\omega t} \tag{3.1}$$

$$B_2 = \bar{B}_2 e^{i\omega t} \tag{3.2}$$

$$B_3 = \bar{B}_3 e^{i\omega t} \tag{3.3}$$

where $\bar{A}$, $\bar{B}_2$ and $\bar{B}_3$ are complex constants.

The time derivatives which occur in the lift expression (1.8), and the moment expression (1.13), can now be calculated. Thus the above three expressions become

$$\frac{\partial A}{\partial t} = i\omega \bar{A} e^{i\omega t}$$

$$\frac{\partial B_2}{\partial t} = i\omega \bar{B}_2 e^{i\omega t}$$

and

$$\frac{\partial B_2}{\partial t} = i\omega \bar{B}_3 e^{i\omega t}$$

Using (3.1), (3.2), and (3.3) respectively, the three time derivatives can be written as

$$\frac{\partial A}{\partial t} = i\omega A$$

$$\frac{\partial B_2}{\partial t} = i\omega B_2$$

$$\frac{\partial B_3}{\partial t} = i\omega B_3$$

Substituting these values in expressions (1.8) and (1.13) of section III-1 they become

$$L = \rho U A + \frac{\rho}{2} i\omega c A - \rho i\omega B_2 \tag{3.4}$$

and

$$M = -\rho U B_2 - \frac{\rho c^2}{16} i\omega A + \frac{\rho}{2} i\omega B_3 \tag{3.5}$$

On substitution of expression (2.23) of section III-2, in the above expression for lift, it becomes

$$L = \rho U A + \frac{\rho}{2} i\omega c A - \rho i\omega \left\{ B_2^\circ + Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_1 \right] \right\}$$

Eliminating λ by means of relation (6.3) section I-6, i.e., $\omega = \lambda U$ the above expression reduces to

$$L = -\rho i\omega B_2^\circ - \rho U A\left[Q_1 + \frac{i\omega c}{U} F_1 \right]$$

Substituting for B_2° expression (2.20) of section III-2 and expression (6.3) of section II-6 for A, it becomes

$$L = -\rho i\omega\left[-\frac{\pi c^2}{4}(\dot{h} + U\alpha) \right] - \frac{\rho U \Gamma^\circ}{F - Q_0 - Q_1}\left[Q_1 + \frac{i\omega c}{U} F_1 \right]$$

which reduces to

$$L = \frac{\pi}{4}\rho i\omega c^2(\dot{h} + U\alpha) + \rho U \Gamma^\circ \frac{Q_1 + \frac{i\omega c}{U} F_1}{Q_0 + Q_1 - F}$$

If expression (5.3) of section II-5 is substituted for Γ°, the above becomes

$$L = \frac{\pi}{4}\rho i\omega c^2(\dot{h} + U\alpha)$$

$$+ \pi\rho c U\left(\dot{h} + U\alpha + \frac{i\omega c}{4}\alpha\right)\frac{Q_1 + \frac{i\omega c}{U}F_1}{Q_0 + Q_1 - F}$$

At this point the symbol $\overline{P_{AR}}$ is introduced, and is defined as

$$\overline{P_{AR}} = \frac{Q_1 + \frac{i\omega c}{U}F_1}{Q_0 + Q_1 - F} \qquad (3.6)$$

Substituting this in the above expression for lift, and rearranging, it can be written in its final form as

$$L = \pi\rho c U\dot{h}\left[\frac{i\omega c}{4U} + \overline{P_{AR}}\right]$$

$$+ \pi\rho c U^2\alpha\left[\frac{i\omega c}{4U} + \left(1 + \frac{i\omega c}{4U}\right)\overline{P_{AR}}\right] \qquad (3.7)$$

The moment expression (3.5) will now be put in final form. For B_2 in this expression substitute formula (2.23) section III-2, and for B_3 substitute formula (2.36) of section III-2. The expression for the moment now becomes

$$M = -\rho U\left\{B_2^\circ + Ac\left[\frac{1}{2} + \frac{1}{i\lambda c} + \frac{Q_1}{i\lambda c} + F_1\right]\right\} - \frac{\rho c^2}{16}i\omega A$$

$$+ \frac{\rho}{2}i\omega\left\{B_3^\circ + Ac^2\left[\frac{1}{8} + \frac{1}{i\lambda c} - \frac{2}{\lambda^2c^2} + \frac{Q_0}{2i\lambda c} - \frac{2Q_1}{\lambda^2c^2} + F_0\right]\right\}$$

Eliminating λ by relation (6.3) section I-6, the expression becomes

$$M = -\rho U B_2^\circ - \rho U A c\left[\frac{1}{2} + \frac{U}{i\omega c} + \frac{UQ_1}{i\omega c} + F_1\right] - \frac{\rho c^2}{16}i\omega A + \frac{\rho}{2}i\omega B_3^\circ$$

$$+ \frac{\rho}{2}i\omega c^2 A\left[\frac{1}{8} + \frac{U}{i\omega c} - \frac{2U^2}{\omega^2c^2} + \frac{UQ_0}{2i\omega c} - \frac{2U^2Q_1}{\omega^2c^2} + F_0\right]$$

which reduced to

$$M = -\rho U B_2^\circ + \frac{\rho}{2} i\omega B_3^\circ + \rho U A c\left[\frac{Q_0}{4} + \frac{i\omega c}{4U} F_0 - F_1\right]$$

Substituting for B_2° expression (2.20) section III-2, for B_3° expression (2.34) section III-2, and for A expression (6.3) section II-6, the above expression becomes

$$M = \frac{\pi}{4}\rho c^2 U\left[\dot{h} + U\alpha\right] + \frac{\pi}{128}\rho\omega^2 c^4\alpha - \rho U \Gamma^\circ c\,\frac{\frac{Q_0}{4} + \frac{i\omega c}{2U} F_0 - F_1}{Q_0 + Q_1 - F}$$

Substituting expression (5.2) section II-5 for Γ°, the above can be written as

$$M = \frac{\pi}{4}\rho c^2 U\left\{\dot{h} + U\alpha + \frac{\omega^2 c^2}{32U}\alpha - \left(\dot{h} + U\alpha + \frac{i\omega c}{4}\alpha\right)\frac{Q_0 + 2\frac{i\omega c}{U}F_0 - 4F_1}{Q_0 + Q_1 - F}\right\}$$

If in the latter fraction one long division is made the above expression can be reduced to

$$M = \frac{\pi}{4}\rho c^2 U\left\{\frac{\omega^2 c^2}{32U}\alpha - \frac{i\omega c}{4}\alpha + \left(\dot{h} + U\alpha + \frac{i\omega c}{4}\alpha\right)\frac{Q_1 - F + 4F_1 - \frac{2i\omega c}{U}F_0}{Q_0 + Q_1 - F}\right\}$$

It is found convenient to symbolize the latter fraction occurring in the above expression; thus

$$\overline{Q_{AR}} = \frac{Q_1 - F + 4F_1 - \frac{2i\omega c}{U}F_0}{Q_0 + Q_1 - F} \qquad (3.8)$$

The moment equation can now be put in its final form as shown below:

$$M = \frac{\pi}{4}\rho c^2 U\dot{h}\,\overline{Q_{AR}} + \frac{\pi}{4}\rho c^2 U^2\alpha\left[\frac{\omega^2 c^2}{32U^2} - \frac{i\omega c}{4U} + \left(1 + \frac{i\omega c}{4U}\right)\overline{Q_{AR}}\right] \qquad (3.9)$$

The quantities $\overline{P}_{AR}$ and $\overline{Q}_{AR}$ are complex functions; hence it is desirable to split them into real and imaginary parts. To do this the following notation is adopted, thus

$$\overline{P}_{AR} = F_{AR} + i G_{AR} \tag{3.10}$$

and

$$\overline{Q}_{AR} = H_{AR} + i J_{AR} \tag{3.11}$$

Appendix A

Relation Between λ , ω, and U

In section I-3, the instantaneous value of the circulation A was introduced. Also in this section the quantities λ , and ω were introduced; the flight velocity U having been previously introduced. In section I-6, the instantaneous nature of the downwash is set forth and the relation between λ , ω, and U is given. Here an explanation of this relation is given.

In figure A 1 is shown the wing chord E F, and also a sine curve with the ξ -axis , (cf. figure 3.1 section I-3). Let this sine curve represent the strength of one of the wake trailing vortices at some particular instant of time. As the wing continues to oscillate

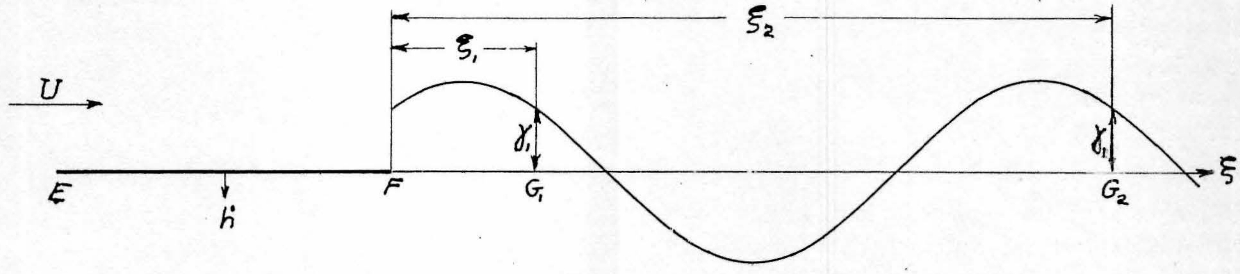

Fig. A.1. - Figure for the Determination of the relation between λ , ω , and U .

the wake trailing vortex is continously generated and flows down stream with the velocity U , retative to the wing. Since the sine curve represents the strength γ_1 , of the trailing vortex, say the real part of γ_1, since it is complex, this sine curve can be thought of as flowing downstream with the velocity U . For a small elapse of time it will move only a small distance; for a larger interval of time it will move a greater distance. For an interval of time equal to a complete period, the sine curve will move downstream just far enough to again coincide with the instantaneous photograph, shown in figure A.I. The ordinate originally at G_1, which is at a distance ξ_1 downstream, is after this particular period of time, at point G_2, a distance ξ_2 downstream.

The time required for this displacement to occur, is equal to the time required for one cycle of the wing's oscillation. The time for one cycle, as found in section I-6, is $T = \frac{2\pi}{\omega}$. The distance between points G_1 and G_2 is then just one wave length of the sine curve. In order to determine this wave length, reference is made to formula (3.1) of section I-3. This formula is

$$\gamma_1 = A e^{-i\lambda\xi}$$

or in trigonometric notation it is

$$\gamma_1 = A(\cos\lambda\xi - i\sin\lambda\xi)$$

Since the values of γ_1 shown at points G_1 and G_2 on figure 1A are equal it follows that $\cos\lambda\xi_1 = \cos\lambda\xi_2$, and $\sin\lambda\xi_1 = \sin\lambda\xi_2$.

Since the $\lambda\xi_1$ is not equal to $\lambda\xi_2$ and just one wave length is involved, it follows from the periodicity of the sine and cosine that

$$\lambda\xi_1 + 2\pi = \lambda\xi_2$$

or

$$\xi_2 - \xi_1 = \frac{2\pi}{\lambda}$$

Consequently from the period T, and the wave length, it is apparent that

$$TU = \frac{2\pi}{\lambda}$$

or

$$\frac{2\pi}{\omega}U = \frac{2\pi}{\lambda}$$

From which it follows that

$$\omega = \lambda U$$

The above may be obtained from a consideration of the fact that the quantity A, occurring in equation (3.1) of section I-3, is an instantaneous value. From this viewpoint A may be written as

$$A = \bar{A}\, e^{i\omega t}$$

where $\bar{A}$ is the complex amplitude of the vortex strength. Substituting in equation (3.1), it becomes

$$\gamma_1 = \bar{A}\, e^{i(\omega t - \lambda \xi)} \qquad (A1)$$

If the point G_1 is flowing with the current, its abscissa ξ will increase with time in such a way that

$$\xi = Ut$$

If the strength of the trailing vortex is to remain constant for the point G_1, as it flows downstream, then the exponent of equation (A 1) must be constant, i.e., in this case

$$\omega t - \lambda \xi = constant$$

or more specifically

$$\omega t - \lambda U t = constant$$

If the above is written in the form

$$(\omega - \lambda U)\, t = constant$$

it is apparent that in order for this to remain constant, for all times, it must be true that

$$\omega = \lambda U$$

hence, the constant is zero.

Appendix B

The Integral $\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau} \cos n\tau \, d\tau$

In section II-2, the integrals (2.1), (2.2), and (2.3) are given together with a recurrence formula, $I_2(\frac{c}{r_1}) = I_0(\frac{c}{r_1}) - \frac{2r_1}{c} I_1(\frac{c}{r_1})$. To outline the procedure of appendix B, the general recurrence formula is first developed, see expression (B 2). With this general recurrence formula, the expression given above, together with two other recurrence formulae, are obtained, see expressions (B 3), (B 4), and (B 5). These formulae are derived without using the theory of Bessel functions. Secondly, the above mentioned integrals are worked out by means of series expensions, and their relationships to the Bessel functions are set forth.

The recurrence formulae are here derived by means of the properties of the integrals. These integrals are each given by the expression

$$\int_0^{\pi} e^{\frac{c}{r_1}\cos\tau} \cos n\tau \, d\tau = \pi I_n(\tfrac{c}{r_1}) \qquad \text{(B 1)}$$

when n is given its proper value. The integral (B 1) is a special case of Hansen's integral and holds for n equal to zero or any integer.

To obtain the recurrence formula consider the two trigonometric identities,

$$\cos(n+1)\tau = \cos n\tau \cos\tau - \sin n\tau \sin\tau$$

and

$$\cos(n-1)\tau = \cos n\tau \cos\tau + \sin n\tau \sin\tau$$

On subtracting these two identities the result becomes

$$\cos(n+1)\tau - \cos(n-1)\tau = -2\sin n\tau \sin\tau$$

or

$$\cos(n+1)\tau = \cos(n-1)\tau - 2\sin n\tau \sin\tau$$

In the latter expression replace n , by $n-1$, and the following trigonometric identity is attained,

$$\cos n\tau = \cos(n-2)\tau - 2\sin(n-1)\tau \sin\tau$$

Substitute this expression for $\cos n\tau$ in the integral (B 1), and it becomes

$$\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau}\left\{\cos(n-2)\tau - 2\sin(n-1)\tau\,\sin\tau\right\}d\tau = \pi I_n\left(\tfrac{c}{\pi_1}\right)$$

Now in integral (B 1) repace n, by $n-2$, then

$$\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau}\cos(n-2)\tau\,d\tau = \pi I_{n-2}\left(\tfrac{c}{\pi_1}\right)$$

hence the former integral becomes

$$\pi I_{n-2}\left(\tfrac{c}{\pi_1}\right) - 2\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau}\sin(n-1)\tau\,\sin\tau\,d\tau = \pi I_n\left(\tfrac{c}{\pi_1}\right)$$

Integrate the remaining integral once by parts. Thus for the parts formula

$\int u\,dv = uv - \int v\,du$, let

$$u = \sin(n-1)\tau, \qquad dv = e^{\frac{c}{\pi_1}\cos\tau}\sin\tau\,d\tau,$$

$$du = (n-1)\cos(n-1)\tau\,d\tau, \qquad v = -\tfrac{\pi_1}{c}e^{\frac{c}{\pi_1}\cos\tau}.$$

Substituting this in the above expression gives

$$\pi I_{n-2}\left(\tfrac{c}{\pi_1}\right) - 2\left\{\left[-\tfrac{\pi_1}{c}e^{\frac{c}{\pi_1}\cos\tau}\sin(n-1)\tau\right]_0^{\pi} + (n-1)\tfrac{\pi_1}{c}\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau}\cos(n-1)\tau\,d\tau\right\} = \pi I_n\left(\tfrac{c}{\pi_1}\right)$$

Since n is a positive integer the second term of the left member is zero, and from (B 1), it is apparent that the integral is $\pi I_{n-1}\left(\tfrac{c}{\pi_1}\right)$; hence

$$\pi I_{n-2}\left(\tfrac{c}{\pi_1}\right) - 2(n-1)\tfrac{\pi_1}{c}\pi I_{n-1}\left(\tfrac{c}{\pi_1}\right) = \pi I_n\left(\tfrac{c}{\pi_1}\right)$$

or

$$I_n\left(\tfrac{c}{\pi_1}\right) = I_{n-2}\left(\tfrac{c}{\pi_1}\right) - 2(n-1)\tfrac{\pi_1}{c}I_{n-1}\left(\tfrac{c}{\pi_1}\right) \tag{B 2}$$

If n is put equal to 2,3, and 4, the three identities used herein can be obtained from (B 2), thus

$$I_2\left(\tfrac{c}{\pi_1}\right) = I_0\left(\tfrac{c}{\pi_1}\right) - \tfrac{2\pi_1}{c}I_1\left(\tfrac{c}{\pi_1}\right) \tag{B 3}$$

$$I_3\left(\tfrac{c}{\pi_1}\right) = -\tfrac{4\pi_1}{c}I_0\left(\tfrac{c}{\pi_1}\right) + \left(1 + \tfrac{8\pi_1^2}{c^2}\right)I_1\left(\tfrac{c}{\pi_1}\right) \tag{B 4}$$

and

$$I_4\left(\frac{c}{\pi_1}\right) = \left(1 + \frac{24\pi_1^2}{C^2}\right) I_0\left(\frac{c}{\pi_1}\right) - \left(\frac{8\pi_1}{C} + \frac{48\pi_1^3}{C^3}\right) I_1\left(\frac{c}{\pi_1}\right) \qquad (B5)$$

If $\frac{c}{\pi_1}$ is replaced by $\frac{c}{12\pi}$ in (B 3) and (B 4) two more required identities are obtained; thus

$$I_2\left(\frac{c}{12\pi_1}\right) = I_0\left(\frac{c}{12\pi_1}\right) - \frac{24\pi_1}{C} I_1\left(\frac{C}{12\pi_1}\right) \qquad (B\ 6)$$

$$I_3\left(\frac{c}{12\pi_1}\right) = -\frac{48\pi_1}{C} I_0\left(\frac{c}{12\pi_1}\right) + \left(1 + 1152\frac{\pi_1^2}{C^2}\right) I_1\left(\frac{c}{12\pi_1}\right) \qquad (B\ 7)$$

It is only necessary to integrate (B 1) for $n = 0$ and 1. For $n = 0$, the integrat becomes

$$\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau} d\tau = \pi I_0\left(\frac{c}{\pi_1}\right) \qquad (B\ 8)$$

This integration is performed by expanding $e^{\frac{c}{\pi_1}\cos\tau}$ in a series and integrating term by term. By this means, a series will be determined for the function $I_0\left(\frac{c}{\pi_1}\right)$. Expanding $e^{\frac{c}{\pi_1}\cos\tau}$ in a power series gives

$$e^{\frac{c}{\pi_1}\cos\tau} = 1 + \frac{c}{\pi_1}\cos\tau + \frac{1}{2!}\frac{C^2}{\pi_1^2}\cos^2\tau + \frac{1}{3!}\frac{C^3}{\pi_1^3}\cos^3\tau + \cdots + \frac{1}{K!}\frac{C^K}{\pi_1^K}\cos^K\tau + \cdots$$

or symbolically

$$e^{\frac{c}{\pi_1}\cos\tau} = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{C}{\pi_1}\right)^K \cos^K\tau$$

Since the above series converges uniformly, for all values of $\frac{c}{\pi_1}$ and τ, it can be integrate term by term. It then follows that

$$\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau} d\tau = \sum_{K=0}^{\infty} \frac{1}{K!}\left(\frac{c}{\pi_1}\right)^K \int_0^{\pi} \cos^K\tau\, d\tau$$

If K is an odd integer, it can be shown that

$$\int_0^{\pi} \cos^K\tau\, d\tau = 0$$

To show this let $\tau = \theta + \frac{\pi}{2}$, then $\theta = -\frac{\pi}{2}$ when $\tau = 0$, and $\theta = \frac{\pi}{2}$ when $\tau = \pi$. Substituting

$$\int_0^{\pi} \cos^K \tau \, d\tau = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^K\left(\theta + \frac{\pi}{2}\right) d\theta$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^K \theta \, d\theta$$

Since the $\sin\theta$ is an odd function with respect to the origin, and since K is odd, $\sin^K\theta$ is an odd function with respect to origin. From the fact, that the integral of an odd function taken over an interval symmetrical with respect to the axis of the ordinates is zero, it follows that

$$\int_0^{\pi} \cos^K \tau \, d\tau = 0$$

Omitting the odd term, the series can now be written as

$$e^{\frac{c}{\pi_1}\cos\tau} = 1 + \frac{1}{2!}\left(\frac{c}{\pi_1}\right)^2 \cos^2\tau + \frac{1}{4!}\left(\frac{c}{\pi_1}\right)^4 \cos^4\tau + \cdots$$

$$+ \frac{1}{(2K)!}\left(\frac{c}{\pi_1}\right)^{2K} \cos^{2K}\tau + \cdots$$

and the integral becomes

$$\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau} d\tau = \sum_{K=0}^{\infty} \frac{1}{(2K)!}\left(\frac{c}{\pi_1}\right)^{2K} \int_0^{\pi} \cos^{2K}\tau \, d\tau \qquad \text{(B 9)}$$

It remains to evaluate

$$\int_0^{\pi} \cos^{2K} \tau \, d\tau$$

Let this integral be designated as

$$c_K = \int_0^{\pi} \cos^{2K} \tau \, d\tau$$

from which it follows that

$$c_{K-1} = \int_0^{\pi} \cos^{2K-2} \tau \, d\tau$$

where K in the first integral has been replaced by $K-1$, and likewise it follows that

$$c_{K-r} = \int_0^{\pi} \cos^{2K-2r} \tau \, d\tau$$

If $K=0$, the integral for c_K becomes

$$c_0 = \int_0^{\pi} d\tau = \pi$$

The integral c_K can be written as

$$c_K = \int_0^{\pi} \cos^{2K-1} \tau \cos\tau \, d\tau$$

Next integrate once by parts; thus in the parts formula $\int u\,dv = uv - \int v\,du$,

let $\quad u = \cos^{2K-1}\tau, \qquad dv = \cos\tau\, d\tau,$

then, $\quad du = -(2K-1)\cos^{2K-2}\tau \sin\tau\, d\tau, \qquad v = \sin\tau,$

hence, $\quad c_K = \cos^{2K-1}\tau \sin\tau \Big]_0^{\pi} + (2K-1)\int_0^{\pi} \cos^{2K-2}\tau \sin^2\tau\, d\tau$

The first term of the right member is zero, and the second can be modified by substituting $1-\cos^2\tau$ for $\sin^2\tau$, from which it follows that

$$\begin{aligned} c_K &= (2K-1)\int_0^{\pi} \cos^{2K-2}\tau (1-\cos^2\tau)\, d\tau \\ &= (2K-1)\int_0^{\pi} \cos^{2K-2}\tau\, d\tau - (2K-1)\int_0^{\pi} \cos^{2K}\tau\, d\tau \\ &= (2K-1)\, c_{K-1} - (2K-1)\, c_K \end{aligned}$$

Solve the above for c_K and the following recurrence formula is obtained; thus

$$c_K = \frac{2K-1}{2K} c_{K-1}$$

By repeated applications of this formula, the integral c_K can be evaluated. To this effect replace K in the above formula by $K-1$ and it becomes

$$c_{K-1} = \frac{2K-3}{2(K-1)} c_{K-2}$$

Substituting this in the formula for C_K, it becomes

$$C_K = \frac{(2K-1)(2K-3)}{2^2 K(K-1)} C_{K-2}$$

continuing in this manner

$$C_K = \frac{(2K-1)(2K-3)(2K-5)}{2^2 K(K-1)(K-2)} C_{K-3}$$

By mathematical induction C_K can be written in terms of C_{K-r} ; thus

$$C_K = \frac{(2K-1)(2K-3)\cdots(2K-2r+1)}{2^r K(K-1)(K-2)\cdots(K-r+1)} C_{K-r}$$

Now let $r = K$ then $C_{r-K} = C_0 = \pi$ and

$$C_K = \frac{(2K-1)(2K-3)\cdots 5\cdot 3\cdot 1}{2^K K(K-1)(K-2)\cdots 3\cdot 2\cdot 1}\pi$$

where π has been substituted for C_0. It will be noticed that the denominator, aside from 2^K, is $K!$, but the numerator is not a factorial number. The numerator can be made $(2K)!$ if it is multiplied by $2K(2K-2)(2K-4)\cdots 4\cdot 2$. If this is done it will be necessary to divide by the same number, but it should be observed that

$$\frac{2K(2K-2)(2K-4)\cdots 4\cdot 2}{2K(2K-2)(2K-4)\cdots 4\cdot 2} = \frac{2K(2K-2)(2K-4)\cdots 4\cdot 2}{2^K[K(K-1)(K-2)\cdots 2\cdot 1]} = 1$$

from which it follows that

$$C_K = \frac{(2K)!}{2^{2K}(K!)^2}\pi = \int_0^\pi \cos^{2K}\tau \, d\tau \qquad \text{(B 10)}$$

If this is substituted in expression (B 9), it becomes

$$\int_0^\pi e^{\frac{c}{\pi_1}\cos\tau} d\tau = \sum_{K=0}^{\infty} \frac{1}{(2K)!}\left(\frac{c}{\pi_1}\right)^2 \left\{\frac{(2K)!}{2^{2K}(K!)^2}\pi\right\}$$

$$= \pi \sum_{K=0}^{\infty} \frac{1}{2^{2K}(K!)^2}\left(\frac{c}{\pi_1}\right)^{2K}$$

On comparing this with (B 8), it is clear that

$$I_0\left(\frac{c}{\pi_1}\right) = \sum_{K=0}^{\infty} \frac{1}{2^{2K}(K!)^2}\left(\frac{c}{\pi_1}\right)^{2K} \qquad \text{(B 11)}$$

Following the outline given in the first paragraph of this appendix, the relation which $I_0(\frac{c}{r_1})$ bears to the Bessel functions, will now be shown. In particular, $I_0(\frac{c}{r_1})$ bears a relation to the Bessel function of the first kind and of zero order. The Bessel function of the first kind and of zero order is written symbolically as $J_0(x)$, where x is the variable concerned.

The usual way to express the Bessel function is by series. The series for the Bessel function of the first kind and of zero order is

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + - \cdots$$

This is given as formula (1.6) p 49 of "Mathematical Methods in Engineering," by Theodore von Kármán and Maurice A. Biot (reference 3). The above expression can be put in the following form

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + - \cdots$$

Now let $x = iy$ where $i = \sqrt{-1}$, then

$$J_0(iy) = 1 + \frac{y^2}{2^2(1!)^2} + \frac{y^4}{2^4(2!)^2} + \frac{y^6}{2^6(3!)^2} + \cdots$$

or, this series can be written as

$$J_0(iy) = \sum_{K=0}^{\infty} \frac{y^{2K}}{2^{2K}(K!)^2}$$

Comparing this series with expression (B 11), it is evident that if $y = \frac{c}{r_1}$, the two are identical, hence

$$J_0\left(i\frac{c}{r_1}\right) = I_0\left(\frac{c}{r_1}\right) \qquad (B\,12)$$

In some texts on Bessel functions the symbol $I_0(y)$ is written as $J_0(iy)$. This is also true in certain tables giving numerical values of Bessel functions; the tables giving $I_0(y)$ are entitled as values of $J_0(iy)$.

The last topic of this appendix concerns integral (B 1) when $n=1$, i.e.,

$$\int_0^\pi e^{\frac{c}{n_1}\cos\tau}\cos\tau\, d\tau = \pi I_1\left(\frac{c}{n_1}\right) \tag{B 13}$$

This is integral (2.2) of section II-2. Here the integrand is expanded in the same way as integral (B 8) was treated; thus

$$e^{\frac{c}{n_1}\cos\tau}\cos\tau = \cos\tau + \left(\frac{c}{n_1}\right)\cos^2\tau + \frac{1}{2!}\left(\frac{c}{n_1}\right)^2\cos^3\tau$$

$$+ \frac{1}{3!}\left(\frac{c}{n_1}\right)^3\cos^4\tau + \cdots + \frac{1}{K!}\left(\frac{c}{n_1}\right)^K\cos^{K+1}\tau + \cdots$$

$$= \sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{c}{n_1}\right)^K\cos^{K+1}\tau$$

This series also converges uniformly and can therefore by integrated term by term; thus

$$\int_0^\pi e^{\frac{c}{n_1}\cos\tau}\cos\tau\, d\tau = \sum_{K=0}^{\infty}\frac{1}{K!}\left(\frac{c}{n_1}\right)^K\int_0^\pi\cos^{K+1}\tau\, d\tau$$

The typical integral in this series is

$$\int_0^\pi\cos^{K+1}\tau\, d\tau$$

This integral is equal to zero if $K+1$ is odd, as was pointed out in the case of integral (B 8). To eliminate the odd powers of $\cos\tau$ in the above series, replace K by $2K+1$, and it becomes

$$\int_0^\pi e^{\frac{c}{n_1}\cos\tau}\cos\tau\, d\tau = \sum_{K=0}^{\infty}\frac{1}{(2K+1)!}\left(\frac{c}{n_1}\right)^{2K+1}\int_0^\pi\cos^{2K+2}\tau\, d\tau$$

The integral to be integrated is now

$$\int_0^\pi\cos^{2K+2}\tau\, d\tau$$

where $K=0,1,2,3,\cdots$. This can be integrated by means of formula (B 10) if K is replaced by $K+1$; thus

$$\int_0^{\pi} \cos^{2K+2}\tau \, d\tau = \frac{(2K+2)!}{2^{2K+2}[(K+1)]^2}\pi$$

Substituting this expression, the desired integral becomes,

$$\int_0^{\pi} e^{\frac{c}{\pi_1}\cos\tau}\cos\tau \, d\tau = \sum_{K=0}^{\infty} \frac{1}{(2K+1)!}\left(\frac{c}{\pi_1}\right)^{2K+1}\left\{\frac{(2K+2)!}{2^{2K+2}[(K+1)!]^2}\pi\right\}$$

$$= \frac{\pi}{2}\left(\frac{c}{\pi_1}\right)\sum_{K=0}^{\infty}\frac{1}{2^{2K}K!(K+1)!}\left(\frac{c}{\pi_1}\right)^{2K} \qquad \text{(B 14)}$$

On comparing with integral (B 13), it follows at once, that

$$I_1\left(\frac{c}{\pi_1}\right) = \frac{1}{2}\left(\frac{c}{\pi_1}\right)\sum_{K=0}^{\infty}\frac{1}{2^{2K}K!(K+1)!}\left(\frac{c}{\pi_1}\right)^{2K} \qquad \text{(B 15)}$$

To show the relationship between $I_1\left(\frac{c}{\pi_1}\right)$ and the Bessel functions, reference is again made to von Kármán and Biot, "Mathematical Methods in Engineering," (see reference 3). On page 57 of this book is found the following formula:

$$J_\nu(x) = \frac{1}{2^\nu \Gamma(\nu+1)} x^\nu \left[1 - \frac{x^2}{1!(\nu+1)2^2} + \frac{x^4}{2!(\nu+1)(\nu+2)2^4} - \cdots\right]$$

This is the series for the Bessel function of the first kind, and of any order. The symbol ν designates the order. Here the interest lies in the Bessel function for which $\nu = 1$. Placing $\nu = 1$, the above expression becomes

$$J_1(x) = \frac{1}{2\Gamma(2)} x \left[1 - \frac{x^2}{1!(2)2^2} + \frac{x^4}{2!(2)(3)2^4} - \cdots\right]$$

Since the value of the gamma function is one, i.e., the $\Gamma(2)=1$, the expression can be written as

$$J_1(x) = \frac{x}{2}\left[1 - \frac{x^2}{2^2 1! 2!} + \frac{x^4}{2^4 2! 3!} - \cdots\right]$$

Now let $x = iy$, and the expression written with its K^{th} term becomes

$$J_1(iy) = \frac{iy}{2}\left[1 + \frac{y^2}{2^2 1!2!} + \frac{y^4}{2^4 2!3!} + \cdots + \frac{y^{2K}}{2^{2K} K!(K+1)!} + \cdots\right]$$

or

$$J_1(iy) = \frac{iy}{2} \sum_{K=0}^{\infty} \frac{y^{2K}}{2^{2K} K!(K+1)!}$$

Multiply both sides of this expression by $-i$, and it then takes the following form:

$$-iJ_1(iy) = \frac{y}{2} \sum \frac{y^{2K}}{2^{2K} K!(K+1)!}$$

On comparing this with series (B 15), it will be seen if $y = \frac{c}{r_1}$, that

$$-iJ_1\left(i\frac{c}{r_1}\right) = I_1\left(\frac{c}{r_1}\right) \qquad \text{(B 16)}$$

Here again it might be mentioned, that there are certain treatises on Bessel functions which do not recognize the symbol $I_1(iy)$, but in its stead use the symbol $-iJ_1(iy)$, and tables are given accordingly. It might also be pointed out, that the results (B 12) and (B 16) can be generalized; thus

$$i^{-\nu} J_\nu(iy) = I_\nu(y)$$

where $\nu =$ the order of the function. This formula is given on page 62 of reference 3.

Appendix C

The Integrals of Section II-3

Before evaluating the integral of section II-3, it is advisable to define the Hankel functions. These are given as equations (3.7) page 59 in "Mathematical Methods in Engineering," (see reference 3). It is however, only the second form which is used here, i.e.,

$$H_\nu^{(2)}(x) = J_\nu(x) - i\,Y_\nu(x)$$

where ν designates the order of the function.

The function $J_\nu(x)$ was explained in appendix B, as the Bessel function of the first kind; in like manner, $Y_\nu(x)$ is called the Bessel function of the second kind. It is also defined as a series, and is a solution to Bessel's differential equation. Here, all that is employed are the orders zero and one; hence

$$H_0^{(2)}(x) = J_0(x) - i\,Y_0(x) \qquad \text{(C 1)}$$

and

$$H_1^{(2)}(x) = J_1(x) - i\,Y_1(x) \qquad \text{(C 2)}$$

To bring the integral of section II-3 into recognizable form, it is necessary to make the following substitution in regards to the variable ξ. To this effect, let

$$\xi = \frac{c}{2}(\varsigma - 1)$$

then

$$d\xi = \frac{c}{2}\,d\varsigma$$

For the limits of integration, it is evident that $\varsigma = 1$ when $\xi = 0$, and $\varsigma = \infty$ when $\xi = \infty$. Using this substitution the integral of section II-3 becomes

$$\int_0^\infty \left[\sqrt{\frac{\xi + c}{\xi}} - 1\right] e^{-i\lambda\xi}\,d\xi = \frac{c\,e^{\frac{i\lambda c}{2}}}{2}\int_1^\infty \left[\frac{\varsigma - 1}{\sqrt{\varsigma^2 - 1}} - 1\right] e^{-\frac{i\lambda c}{2}\varsigma}\,d\varsigma \qquad \text{(C 3)}$$

The integral on the right will be treated as two integrals, separated as follows:

$$\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\sqrt{\varsigma^2-1}}\,d\varsigma \qquad \text{(C 4)}$$

and

$$\int_1^\infty \left[\frac{\varsigma}{\sqrt{\varsigma^2-1}}-1\right] e^{-\frac{i\lambda c}{2}\varsigma}\,d\varsigma \qquad \text{(C 5)}$$

An integral which evaluates (C 4) will now be constructed from integrals XIV and XV, of Theodore Theodorsen's "General Theory of Aerodynamic Instability and the Mechanism of Flutter"; T.R. No. 496 N.A.C.A. 1940 (reference 4). Multiplying the first of these integrals by i and adding to the second, they become

$$\int_1^\infty \left[i\frac{\cos kx}{\sqrt{x^2-1}} + \frac{\sin kx}{\sqrt{x^2-1}}\right]dx = \frac{\pi}{2}\left[J_0(k) - i\,Y_0(k)\right]$$

By means of expression (C 1), and from the fact that $\cos kx - i\sin kx = e^{-ikx}$, the above integral can be put in the following form, i.e.,

$$\int_1^\infty \frac{e^{-ikx}}{\sqrt{x^2-1}}\,dx = -\frac{\pi}{2} i H_0^{(2)}(k) \qquad \text{(C 6)}$$

On comparing this with integral (C 4) it is evident that

$$\int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\sqrt{\varsigma^2-1}}\,d\varsigma = -\frac{\pi}{2} i H_0^{(2)}\left(\frac{\lambda c}{2}\right) \qquad \text{(C 7)}$$

The method which follows concerning integral (C 5) is given in an article written by Theodore von Kármán and William R. Sears, entitled "Airfoil Theory for Non-Uniform Motion," see reference 5.

In order to build up an integral which will correspond to (C 5), consider the identity written with the help of integral (C 6), i.e.,

$$\int_1^\infty \left[\frac{e^{-ikx}}{\sqrt{x^2-1}} - \frac{e^{-ikx}}{x}\right]dx = -\frac{\pi}{2} i H_0^{(2)}(k) - \int_1^\infty \frac{e^{-ikx}}{x} dx$$

The integrals being convergent, the next step is to differentiate with respect to k. The left side of the above expression yields a convergent integral when differentiated, while the integral on the right side becomes divergent, it is bounded in absolute value. To make this integral converge after differentiation, a new variable t is introduced. Let $t = kx$, then $dt = kdx$. For the limits of integration $t=k$ when $x=1$ and $t=\infty$ when $x=\infty$; hence

$$\int_1^\infty \left[\frac{1}{\sqrt{x^2-1}} - \frac{1}{x}\right] e^{-ikx} dx = -\frac{\pi}{2} i H_0^{(2)}(k) - \int_k^\infty \frac{e^{-it}}{t} dt$$

Replace $H_0^{(2)}(k)$ by its equivalent as given by expression (C 1), and differentiate both sides of the above expression with respect to k as follows:

$$-i\int_1^\infty \left[\frac{x}{\sqrt{x^2-1}} - 1\right] e^{-ikx} dx = -\frac{\pi}{2} i \left[J_0'(k) - iY_0'(k)\right] + \frac{e^{-ik}}{k}$$

where the primes on the Bessel functions designate differentiation with respect to k. Dividing this expression by $-i$ it becomes

$$\int_1^\infty \left[\frac{x}{\sqrt{x^2-1}} - 1\right] e^{-ikx} dx = \frac{\pi}{2} \left[J_0'(k) - iY_0'(k)\right] - \frac{e^{-ik}}{ik}$$

The derivatives of the Bessel functions of order zero are given as equations (4.2), page 60, in "Mathematical Methods in Engineering," (reference

3), and are

$$-\frac{d}{dx}J_0(x) = J_1(x)$$

$$-\frac{d}{dx}Y_0(x) = Y_1(x)$$

hence

$$J_0'(k) - iY_0'(k) = -[J_1(x) - iY_1(x)]$$

By expression (C 2), the above equals $-H_1^{(2)}(x)$. Using this result the above integral can be written as

$$\int_1^\infty \left[\frac{x}{\sqrt{x^2-1}} - 1\right] e^{-ikx} dx = -\frac{e^{-ik}}{ik} - \frac{\pi}{2} H_1^{(2)}(k) \qquad \text{(C 8)}$$

On comparing this with integral (C 5), it will be observed that

$$\int_1^\infty \left[\frac{\xi}{\sqrt{\xi^2-1}} - 1\right] e^{-\frac{i\lambda c}{2}\xi} d\xi = -\frac{2e^{-\frac{i\lambda c}{2}}}{i\lambda c} - \frac{\pi}{2} H_1^{(2)}\left(\frac{\lambda c}{2}\right) \qquad \text{(C 9)}$$

Since the above integral converges, it represents the function on the right hand side.

Using results (C 6) and (C 9), integral (C 3) can be put in its final form; thus

$$\int_0^\infty \left[\sqrt{\frac{\xi+c}{\xi}} - 1\right] e^{-i\lambda\xi} d\xi = -\frac{1}{i\lambda}\Big[1 + \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda c}{2}} H_0^{(2)}\left(\frac{\lambda c}{2}\right) + \frac{\pi}{4} i\lambda c\, e^{\frac{i\lambda c}{2}} H_1^{(2)}\left(\frac{\lambda c}{2}\right)\Big] \qquad \text{(C 10)}$$

Appendix D

Details of Expressions Occurring in Lift and Moment Equations

Von Kármán and Burgers show on page 46 of reference 2, that the vorticity γ on the wing is

$$\gamma = -\frac{V}{\pi \sin\theta}\int_0^{2\pi}\frac{dy}{dx}\cot\frac{\theta-\tau}{2}\sin\tau\, d\tau + \frac{\Gamma}{2\pi a \sin\theta}$$

In particular this ~~is~~ equation (9.20) in the above reference. In section II-1 are given the following formulae:

$$w = V\frac{dy}{dx}$$

and

$$a = \frac{c}{4}$$

Substituting these in the expression for γ, gives

$$\gamma = -\frac{1}{\pi \sin\theta}\int_0^{2\pi} w \sin\tau \cot\frac{\theta-\tau}{2}\, d\tau + \frac{2\Gamma}{\pi c \sin\theta}$$

where w is the downwash, and c is the chord of the wing.

The integration must be carried out with respect to τ from 0 to 2π. It is more convenient to integrate from 0 to π rather than 0 to 2π. To do this the integral is split in the following manner; thus

$$\gamma = -\frac{1}{\pi \sin\theta}\left\{\int_0^{\pi} w \sin\tau \cot\frac{\theta-\tau}{2}\, d\tau + \int_{-\pi}^{0} w \sin\tau \cot\frac{\theta-\tau}{2}\, d\tau\right\} + \frac{2\Gamma}{\pi c \sin\theta}$$

In the second integral of the above expression, let $\tau = -\tau'$ then $\tau' = 0$ when $\tau = 0$, and $\tau' = \pi$ when $\tau = -\pi$. Substituting

$$\int_{-\pi}^{0} w \sin\tau \cot\frac{\theta-\tau}{2}\, d\tau = \int_{\pi}^{0} w \sin(-\tau') \cot\frac{\theta+\tau'}{2}\, d(-\tau')$$

$$= -\int_0^{\pi} w \sin\tau' \cot\frac{\theta+\tau'}{2}\, d\tau'$$

Both integrals are now to the same limits; hence the primes can be dropped and the expression for γ can now be written as

$$\gamma = -\frac{1}{\pi \sin\theta}\int_0^{\pi} w \sin\tau\left[\cot\frac{\theta-\tau}{2} - \cot\frac{\theta+\tau}{2}\right] d\tau + \frac{2\Gamma}{\pi c \sin\theta}$$

The above can be further simplified for it can be shown by means of the trigonometric identities that

$$\cot\frac{\theta-\tau}{2} - \cot\frac{\theta+\tau}{2} = -\frac{2\sin\tau}{\cos\theta - \cos\tau}$$

hence

$$\gamma = \frac{2}{\pi\sin\theta}\int_0^{\pi}\frac{w \sin^2\tau}{\cos\theta - \cos\tau}d\tau + \frac{2\Gamma}{\pi c\sin\theta} \qquad (1\text{ D})$$

To obtain integral (1.7) section III-1, let $x = \frac{c}{2}\cos\theta$ which relation is taken from page 35 formula (6.4) of von Kármán and Burgers' work, here given as reference 2, then equation (1.5) section III-1 becomes

$$B_2 = \frac{c^2}{4}\int_0^{\pi}\gamma\cos\theta\sin\theta\, d\theta$$

Substitute expression (1 D) for γ then

$$B_2 = \frac{c^2}{4}\int_0^{\pi}\left[\frac{2}{\pi\sin\theta}\int_0^{\pi}\frac{w\sin^2\tau}{\cos\theta-\cos\tau}d\tau + \frac{2\Gamma}{\pi c\sin\theta}\right]\cos\theta\sin\theta\, d\theta$$

The above can be written as

$$B_2 = \frac{c^2}{2\pi}\int_0^{\pi}\int_0^{\pi}\frac{w\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}d\theta\, d\tau + \frac{c\Gamma}{2\pi}\int_0^{\pi}\cos\theta\, d\theta$$

If the order of integration be reversed in the first integral the integration with respect to θ can be made at once; thus

$$\int_0^{\pi}\frac{w\sin^2\tau\cos\theta}{\cos\theta-\cos\tau}d\theta = w\sin^2\tau\int_0^{\pi}\frac{\cos\theta\, d\theta}{\cos\theta-\cos\tau}$$

This integral is one form of a more general type, which is worked out in detail by von Kármán and Burgers, in reference 2, on pages 173 and 174 and is there given as

$$\int_0^{\pi} \frac{\cos n\psi'}{\cos\psi' - \cos\psi}\, d\psi' = \pi \frac{\sin n\psi}{\sin\psi} \tag{2 D}$$

Here ψ' corresponds to θ , ψ corresponds to τ , and $n = 1$ in the expression for B_2; hence by formula (2 D)

$$\int_0^{\pi} \frac{w \sin^2\tau \cos\theta}{\cos\theta - \cos\tau}\, d\theta = w \sin^2\tau\,[\pi]$$

The second integral occurring in B_2, as given above is zero, so that it becomes

$$B_2 = \frac{c^2}{2}\int_0^{\pi} w \sin^2\tau\, d\tau \tag{3 D}$$

In order to transform integral (1.10) of section III-1 to that given by expression (1.11), use the relation $x = \frac{c}{2}\cos\theta$, and formula (1 D) With these, it becomes

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx = \int_0^{\pi} \left\{ \frac{2}{\pi \sin\theta} \int_0^{\pi} \frac{w \sin^2\tau}{\cos\theta - \cos\tau}\, d\tau + \frac{2\Gamma}{\pi c \sin\theta} \right\} \frac{c^3}{8} \cos^2\theta \sin\theta\, d\theta$$

$$= \frac{c^3}{4\pi}\int_0^{\pi} \cos^2\theta \int_0^{\pi} \frac{w \sin^2\tau}{\cos\theta - \cos\tau}\, d\tau\, d\theta + \frac{c^2\Gamma}{4\pi}\int_0^{\pi} \cos^2\theta\, d\theta$$

For the first integral of the right member the order of integration is again reversed, and since $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$, the integral can be written as

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx = \frac{c^3}{8\pi}\int_0^{\pi} w \sin\tau \int_0^{\pi} \frac{1 + \cos 2\theta}{\cos\theta - \cos\tau}\, d\theta\, d\tau + \frac{c^2\Gamma}{8\pi}\int_0^{\pi} (1 + \cos 2\theta)\, d\theta$$

In the first integral the part concerning θ can be expressed as follows; thus

$$\int_0^{\pi} \frac{1 + \cos 2\theta}{\cos\theta - \cos\tau}\, d\theta = \int_0^{\pi} \frac{d\theta}{\cos\theta - \cos\tau} + \int_0^{\pi} \frac{\cos 2\theta}{\cos\theta - \cos\tau}\, d\theta$$

Comparing this with integral (2 D) it is clear, that for the first integral of the right member $n=0$, and for the second $n=2$; hence

$$\int_0^{\pi} \frac{1+\cos 2\theta}{\cos\theta-\cos\tau}\,d\theta = 0 + \pi\,\frac{\sin 2\tau}{\sin\tau}$$

Integral (1.10) section III-1 now becomes

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx = \frac{c^3}{8}\int_0^{\pi} w \sin\tau \sin 2\tau \, d\tau + \frac{c^2\Gamma}{8}$$

and since $\sin 2\tau = 2\sin\tau\cos\tau$ it can be written as

$$\int_{-\frac{c}{2}}^{\frac{c}{2}} \gamma x^2 dx = \frac{c^3}{4}\int_0^{\pi} w \sin^2\tau \cos\tau \, d\tau + \frac{c^2\Gamma}{8} \qquad (4\ D)$$

Appendix E

Integrals Occurring in Lift and Moment Equations

The integral of expression (2.15) section III-2, is transformed by the same substitution as is used in appendix C, viz.,

$$\xi = \frac{c}{2}(\varsigma - 1)$$

Applying this substitution to the above mentioned integral, gives

$$\int_0^\infty \left[\frac{2\xi}{c} + 1 - \sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1} \right] e^{-i\lambda\xi} d\xi$$

$$= \frac{c}{2} e^{\frac{i\lambda c}{2}} \int_1^\infty \left[\varsigma - \sqrt{\varsigma^2 - 1} \right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma \qquad (1E)$$

To integrate the above, consider integral (C 9) of appendix C. This integral is

$$\int_1^\infty \left[\frac{\varsigma}{\sqrt{\varsigma^2-1}} - 1 \right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma = -\frac{2e^{-\frac{i\lambda c}{2}}}{i\lambda c} - \frac{\pi}{2} H_1^{(2)}\left(\frac{\lambda c}{2}\right)$$

Integrate once by parts as follows.

If $\int u\,dv = uv - \int v\,du$ is the parts formula

let $u = e^{-\frac{i\lambda c}{2}\varsigma}$ and $dv = \left[\frac{\varsigma}{\sqrt{\varsigma^2-1}} - 1\right] d\varsigma$

then $du = -\frac{i\lambda c}{2} e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma$ and $v = \sqrt{\varsigma^2-1} - \varsigma$

From this it follows that

$$\int_1^\infty \left[\frac{\varsigma}{\sqrt{\varsigma^2-1}} - 1 \right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma = \left[\left(\sqrt{\varsigma^2-1} - \varsigma\right) e^{-\frac{i\lambda c}{2}\varsigma} \right]_1^\infty$$

$$+ \frac{i\lambda c}{2} \int_1^\infty \left[\sqrt{\varsigma^2-1} - \varsigma \right] e^{-\frac{i\lambda c}{2}} d\varsigma$$

The value of the first term of the right member in the above expression is obviously $-e^{-\frac{i\lambda c}{2}}$ when $\varsigma = 1$ but when $\varsigma = \infty$ its value is not obvious. In $\sqrt{\varsigma^2-1}$ factor out a ς ; thus

$$\sqrt{\zeta^2-1} = \zeta\sqrt{1-\frac{1}{\zeta^2}}$$

The radical on the right side has an expansion about the point at infinity which is

$$\sqrt{1-\frac{1}{\zeta^2}} = 1 - \frac{1}{2\zeta^2} - \frac{1}{8\zeta^4} - \cdots$$

From this it follows that

$$\sqrt{\zeta^2-1} - \zeta = \zeta\left(1-\frac{1}{2\zeta^2}-\frac{1}{8\zeta^4}-\cdots\right) - \zeta$$

$$= -\frac{1}{2\zeta} - \frac{1}{8\zeta^3} - \cdots$$

It is now apparent that

$$\underset{\zeta\to\infty}{\mathrm{LIMIT}}\left(\sqrt{\zeta^2-1}-\zeta\right) = \underset{\zeta\to\infty}{\mathrm{LIMIT}}\left(-\frac{1}{2\zeta}-\frac{1}{8\zeta^3}-\cdots\right)$$
$$=0$$

The other factor in the above expression is $e^{-\frac{i\lambda c}{2}\zeta}$. Since ζ is real, the absolute value of this factor remains finite as ζ tends to infinity, as a matter of fact

$$\left|e^{-\frac{i\lambda c}{2}\zeta}\right| = 1$$

for all real values of ζ. It can now be concluded, that

$$\underset{\zeta\to\infty}{\mathrm{LIMIT}}\left(\sqrt{\zeta^2-1}-\zeta\right)e^{-\frac{i\lambda c}{2}\zeta} = 0$$

The above integral can now be written as

$$\int_1^\infty\left[\frac{\zeta}{\sqrt{\zeta^2-1}}-1\right]e^{-\frac{i\lambda c}{2}\zeta}d\zeta = e^{-\frac{i\lambda c}{2}}$$

$$+\frac{i\lambda c}{2}\int_1^\infty\left[\sqrt{\zeta^2-1}-\zeta\right]e^{-\frac{i\lambda c}{2}\zeta}d\zeta$$

On transposing and dividing by $\frac{i\lambda c}{2}$, the above becomes

$$\int_1^\infty \left[\zeta - \sqrt{\zeta^2-1}\right] e^{-\frac{i\lambda c}{2}\zeta} d\zeta = \frac{2e^{-\frac{i\lambda c}{2}}}{i\lambda c} - \frac{2}{i\lambda c}\int_1^\infty \left[\frac{\zeta}{\sqrt{\zeta^2-1}} - 1\right] e^{-\frac{i\lambda c}{2}\zeta} d\zeta$$

Substituting for the integral in the right member, its value is given by expression (C 9) of appendix C, the above expression gives the following result:

$$\int_1^\infty \left[\zeta - \sqrt{\zeta^2-1}\right] e^{-\frac{i\lambda c}{2}\zeta} d\zeta = \frac{2e^{-\frac{i\lambda c}{2}}}{i\lambda c} - \frac{4e^{-\frac{i\lambda c}{2}}}{\lambda^2 c^2} + \frac{\pi}{i\lambda c} H_1^{(2)}\left(\frac{\lambda c}{2}\right) \qquad \text{(E 2)}$$

From this integral (E 1) can be written as

$$\int_0^\infty \left[\frac{2\xi}{c} + 1 - \sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1}\right] e^{-i\lambda\xi} d\xi = \frac{1}{i\lambda} - \frac{2}{\lambda^2 c^2} + \frac{\pi}{2}\frac{e^{\frac{i\lambda c}{2}}}{i\lambda} H_1^{(2)}\left(\frac{\lambda c}{2}\right) \qquad \text{(E 3)}$$

For the integral occurring in expression (2.28), the same substitution will be used as in the preceding paragraphs, i.e.,

$$\xi = \frac{c}{2}(\zeta - 1)$$

and

$$d\xi = \frac{c}{2} d\zeta$$

With this, the following can be written

$$\int_0^\infty \left\{ \frac{1}{2} + \left[\left(\frac{2\xi}{c}+1\right)^2 - 1 \right] - \left(\frac{2\xi}{2}+1\right)\sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1} \right\} e^{-i\lambda\xi}\, d\xi$$

$$= \frac{c}{2} e^{\frac{i\lambda c}{2}} \int_1^\infty \left[\varsigma^2 - \frac{1}{2} - \varsigma\sqrt{\varsigma^2 - 1} \right] e^{-\frac{i\lambda c}{2}\varsigma}\, d\varsigma \qquad \text{(E 4)}$$

To begin, an integration by parts will be performed. Using the parts formula $\int u\,dv = uv - \int v\,du$ let

$$u = \varsigma^2 - \frac{1}{2} - \varsigma\sqrt{\varsigma^2-1}, \qquad dv = e^{-\frac{i\lambda c}{2}\varsigma}\, d\varsigma$$

$$du = \left[2\varsigma - \frac{\varsigma^2}{\sqrt{\varsigma^2-1}} - \sqrt{\varsigma^2-1} \right] d\varsigma, \qquad v = -\frac{2e^{-\frac{i\lambda c}{2}\varsigma}}{i\lambda c}$$

The integral of the right hand side of the above expression becomes, if the prefactor is omitted,

$$\int_1^\infty \left[\varsigma^2 - \frac{1}{2} - \varsigma\sqrt{\varsigma^2-1} \right] e^{-\frac{i\lambda c}{2}\varsigma}\, d\varsigma$$

$$= \left[\left(\varsigma^2 - \frac{1}{2} - \varsigma\sqrt{\varsigma^2-1} \right) \left(-\frac{2e^{-\frac{i\lambda c}{2}\varsigma}}{i\lambda c} \right) \right]_1^\infty$$

$$+ \frac{2}{i\lambda c} \int_1^\infty \left[2\varsigma - \frac{\varsigma^2}{\sqrt{\varsigma^2-1}} - \sqrt{\varsigma^2-1} \right] e^{-\frac{i\lambda c}{2}\varsigma}\, d\varsigma$$

The uv term of the parts formula will be discussed first. It will be shown that the first bracketed term is zero, i.e.,

$$\underset{\varsigma \to \infty}{\text{LIMIT}} \left(\varsigma^2 - \frac{1}{2} - \varsigma\sqrt{\varsigma^2-1} \right) = 0$$

To show this, follow the method which was used in the case of integral (E 1), given in the preceding paragraphs. As was shown

$$\sqrt{1-\frac{1}{\zeta^2}} = 1 - \frac{1}{2\zeta^2} - \frac{1}{8\zeta^4} - \cdots$$

The above expression can be written as

$$\left(\zeta^2 - \frac{1}{2} - \zeta\sqrt{\zeta^2-1}\right) = \zeta^2 - \frac{1}{2} - \zeta^2\left(1 - \frac{1}{2\zeta^2} - \frac{1}{8\zeta^4} - \cdots\right)$$

$$= \zeta^2 - \frac{1}{2} - \zeta^2 + \frac{1}{2} + \frac{1}{8\zeta^2} + \cdots$$

$$= \frac{1}{8\zeta^2} + \cdots$$

hence

$$\underset{\zeta\to\infty}{\text{LIMIT}}\left(\zeta^2 - \frac{1}{2} - \zeta\sqrt{\zeta^2-1}\right) = \underset{\zeta\to\infty}{\text{LIMIT}}\left(\frac{1}{8\zeta^2} + \cdots\right) = 0$$

Also in the preceding discussion it was shown that

$$\left|e^{-\frac{i\lambda c}{2}\zeta}\right| = 1$$

for all real ζ . From this it follows that

$$\underset{\zeta\to\infty}{\text{LIMIT}}\left[\left(\zeta^2 - \frac{1}{2} - \zeta\sqrt{\zeta^2-1}\right)\left(-\frac{2e^{-\frac{i\lambda c}{2}\zeta}}{i\lambda c}\right)\right] = 0$$

In the above expression at the lower limit, i.e. for $\zeta = 1$, the value becomes

$$-\frac{e^{-\frac{i\lambda c}{2}}}{i\lambda c}$$

and hence the above integral becomes

$$\int_1^\infty \left[\zeta^2 - \frac{1}{2} - \zeta\sqrt{\zeta^2-1}\right] e^{-\frac{i\lambda c}{2}\zeta} d\zeta = \frac{e^{-\frac{i\lambda c}{2}}}{i\lambda c}$$

$$+ \frac{2}{i\lambda c}\int_1^\infty \left[2\zeta - \frac{\zeta^2}{\sqrt{\zeta^2-1}} - \sqrt{\zeta^2-1}\right] e^{-\frac{i\lambda c}{2}\zeta} d\zeta \qquad (E5)$$

The integral on the right side can be separated by adding one and subtracting one in the numerator of the second term; thus

$$\int_1^\infty \left[2\varsigma - \frac{\varsigma^2}{\sqrt{\varsigma^2-1}} - \sqrt{\varsigma^2-1}\right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma$$

$$= \int_1^\infty \left[2\varsigma - \frac{\varsigma^2-1+1}{\sqrt{\varsigma^2-1}} - \sqrt{\varsigma^2-1}\right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma$$

$$= \int_1^\infty \left[2(\varsigma - \sqrt{\varsigma^2-1}) - \frac{1}{\sqrt{\varsigma^2-1}}\right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma$$

This latter expression can be separated into two integrals both of which converge. Further, both of the resulting integrals have been evaluated. Performing this operation, the integral becomes

$$\int_1^\infty \left[2\varsigma - \frac{\varsigma^2}{\sqrt{\varsigma^2-1}} - \sqrt{\varsigma^2-1}\right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma$$

$$= 2\int_1^\infty (\varsigma - \sqrt{\varsigma^2-1}) e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma - \int_1^\infty \frac{e^{-\frac{i\lambda c}{2}\varsigma}}{\sqrt{\varsigma^2-1}} d\varsigma$$

The first integral is given as (E 2) of this appendix, and the second by (C 7) appendix C. Substituting (E 2) and (C 7), the above integral becomes

$$\int_1^\infty \left[2\varsigma - \frac{\varsigma^2}{\sqrt{\varsigma^2-1}} - \sqrt{\varsigma^2-1}\right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma$$

$$= \frac{4e^{-\frac{i\lambda c}{2}}}{i\lambda c} - \frac{8e^{-\frac{i\lambda c}{2}}}{\lambda^2 c^2} + \frac{2\pi}{i\lambda c} H_1^{(2)}\left(\frac{\lambda c}{2}\right) + \frac{\pi}{2} i H_0^{(2)}\left(\frac{\lambda c}{2}\right)$$

Substituting this in integral (E 5), gives

$$\int_1^\infty \left[\varsigma^2 - \frac{1}{2} - \varsigma\sqrt{\varsigma^2-1}\right] e^{-\frac{i\lambda c}{2}\varsigma} d\varsigma = \frac{e^{-\frac{i\lambda c}{2}}}{i\lambda c} - \frac{8e^{-\frac{i\lambda c}{2}}}{\lambda^2 c^2}$$

$$- \frac{16e^{-\frac{i\lambda c}{2}}}{i\lambda^3 c^3} - \frac{4\pi}{\lambda^2 c^2} H_1^{(2)}\left(\frac{\lambda c}{2}\right) + \frac{\pi}{i\lambda c} i H_0^{(2)}\left(\frac{\lambda c}{2}\right)$$

(E 6)

Substituting this result in integral (E 4) it becomes

$$\int_0^\infty \left\{ \frac{1}{2} + \left[\left(\frac{2\xi}{c}+1\right)^2 - 1\right] - \left(\frac{2\xi}{c}+1\right)\sqrt{\left(\frac{2\xi}{c}+1\right)^2 - 1} \right\} e^{-i\lambda c} d\xi$$

$$= \frac{1}{2i\lambda} - \frac{4}{\lambda^2 c} - \frac{8}{i\lambda^3 c^2} + \frac{\pi}{2i\lambda} e^{\frac{i\lambda c}{2}} i H_0^{(2)}\left(\frac{\lambda c}{2}\right)$$

$$+ \frac{2\pi}{\lambda^2 c} e^{\frac{i\lambda c}{2}} H_1^{(2)}\left(\frac{\lambda c}{2}\right) \qquad \text{(E 7)}$$